A Modern Approach to Born Reciprocity

by

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Declaration

This thesis contains no material which has been accepted for the award of any other higher degree or graduate diploma in any tertiary institution. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference has been made in the text of the thesis.

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Abstract

In the early twentieth century, Max Born attempted to develop a theory he called the "principle of reciprocity". He observed that many formulae of physics remain unchanged under the following transformation of space-time coordinates and momentum-energy variables: $x^\mu \to p^\mu$; $p^\mu \to -x^\mu$. Examples include Hamilton's equations and Heisenberg's commutation relations. Born's attempts to expand this observation to a general theory were largely unsuccessful. More recently, Stephen Low has made use of the group theoretical methods of Eugene Wigner, Valentine Bargmann and George Mackey to study a group which possesses Born reciprocity as an intrinsic symmetry, called the "quaplectic group", $\mathcal{Q}(1,3)$. This involves the postulation of a new fundamental constant: the maximum rate of change of momentum (or maximum force) - denoted by b. It also involves a new space-time-momentum-energy line element:

$$ds^2 = -dt^2 + \frac{1}{c^2}dx^2 + \frac{1}{b^2}(dp^2 - \frac{1}{c^2}de^2),$$

which remains invariant under quaplectic transformations.

In this work we consider the different contraction limits of the quaplectic group in analogy to the contraction of the Poincaré group to the Galilei group in the limit $c \to \infty$. In particular, the quaplectic group contracts to the Poincaré group in the limit $b \to \infty$ (under the constraint that the reference frames must be inertial) and to the Hamilton group - the group of non-inertial classical mechanics - in the limit $b, c \to \infty$. For the compact group $\mathcal{Q}(2)$ acting on two dimensional Euclidean space we consider the branching rules and use the P^2 Casimir operator of the Euclidean subgroup to label states. We consider the implications of Born reciprocity to the Schrödinger-Robertson inequality, concluding that the covariance matrix Σ is quaplectic invariant and that physically distinct semi-classical limits of two different but unitarily-equivalent minimal uncertainty states must be related by a unitary transformation which does not belong to the quaplectic group. Finally, we explore the worldline quantisation of a system invariant under reciprocal relativity, finding that the square of the energy-momentum tensor is continuous over the entire real line. The resulting states therefore include tachyonic and null states as well as massless states of continuous spin which cannot be projected out in the current formulation. These states are discussed along with the massive states.

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Finally, I dedicate this work to my beautiful wife, Natalie.

Contents

Chapter 1

Introduction

> "I present here an idea which seems to be attractive by its simplicity and may lead to a satisfactory theory."

These are the words Max Born used to finish the introduction to his first paper on the subject of reciprocity, *A suggestion for unifying quantum theory and relativity* [6]. Born wrote several more papers on this topic along with his students such as Herbert S. Green, culminating in the 1949 paper *Reciprocity theory of elementary particles* [8]. In the latter half of the 20th century the theory became less popular, although there have been some notable attempts to develop Born's theory. In particular, Stephen Low has used the group theoretical methods of Wigner, Bargmann and Mackey to develop a symmetry group which is reciprocally invariant. This group has been called the "quaplectic group" by Low, denoted by $\mathcal{Q}(1,n)$ where n is the number of spatial and momentum coordinates. These aspects of Born reciprocity are discussed in Chapter 3.

The symmetry group of special relativistic physics is the Poincaré group. From purely theoretical concepts, the mathematics of the Poincaré group gives rise to such physical concepts as mass and a discrete mass spectrum, spin, special relativity, conservation of angular momentum and so on. But there are several important symmetries and results that do not arise naturally from the Poincaré group. For example, the canonical commutation relations - ubiquitous to quantum mechanics - must be added onto the theory. There are also many unsolved problems in physics which the Poincaré group is unable to solve from group theoretical concepts. It is possible therefore - indeed, quite likely - that there is a deeper symmetry group underlying reality. The quaplectic group is a candidate for this deeper symmetry group. It has the Heisenberg symmetry of the canonical commutation relations built into the structure of the group. It also has the Poincaré group as a subgroup (actually, it has four Poincaré type subgroups; one for each of space-time, momentum-energy, space-energy and momentum-time). It is a higher-dimensional theory, but it does not have to explain why we cannot observe the extra dimensions (we *can* observe them - they are energy and momentum). And, it incorporates the symmetry of Born reciprocity.

If the quaplectic group is indeed this deeper symmetry group of reality, then well known group theoretical concepts can be applied to the group in order to determine if it can solve some of the unsolved problems in physics such as the

galactic rotation problem, quantum gravity and the grand unification theory or the associated theory of everything. The motivation of the current work is to investigate several aspects of the quaplectic group in order to shed further light on whether or not this group could possibly be used for investigating these problems.

1.1 Outline

This work is ordered as follows. By way of introduction, a brief biography of Born is given in Section 1.2. The notation used throughout the rest of this thesis is then summarised in Section 1.3, and an introduction to Born reciprocity, along with a discussion of the work performed by others on the topic, in particular, Herbert S. Green and Eduardo R. Caianiello, is given in Section 1.4.

Chapter 2 presents some background work on group representations. Semidirect product groups and the Mackey method of induced representations are studied in some depth. In particular, the method is explicitly set out for the important case of non-abelian normal subgroups, which is not the usual case used by Wigner [81]. This is essential in deriving the representations of the quaplectic group since the normal subgroup is the non-abelian Weyl-Heisenberg group. The chapter then proceeds to discuss some of the most important groups in theoretical physics: the translation group, the Weyl-Heisenberg group and the Poincaré group. The Mackey method of induced representations is used to calculate the representations of each of these groups as examples of how to use the method. The translation group is a direct product of abelian groups (which are also translation groups). The stabiliser group is the whole translation group, so the Mackey method of induced representations does not need to be used and the representations are just the direct product of the representations of the abelian subgroups. The Weyl-Heisenberg group has two little groups: a classical case - which is just a translation group - and a quantum case. For the quantum case, the stabiliser group is not the whole Weyl-Heisenberg group, and so the Mackey method must be used to obtain the representations. Similarly, for the Poincaré group (or the inhomogeneous Lorentz group), there are four (or six) subgroups, none of which have the Poincaré group as the stabiliser group and so the Mackey method must be used to obtain the representations. This is done for the timelike and massless cases. These groups are also important as background for the rest of this thesis. The chapter ends with a discussion of the field equations of the Poincaré group and how these can be used to derive, for example, the Klein-Gordon equation.

Chapter 3 reviews the work of Low and some others on the topic of Born reciprocity. The representations of the quaplectic group are derived using the Mackey method of induced representations for non-abelian normal subgroups - although there is no induction necessary as the stabiliser group is the whole quaplectic group. The quaplectic algebra and Casimir operators are also studied. Quaplectic symmetry suggests a new kind of relativity, called "reciprocal relativity", and this is studied in some depth. Most of the work in this chapter is due to Low, except for the derivation of time dilation under reciprocal relativity, which is original but unpublished. The chapter ends with a discussion of Rinder acceleration under

reciprocal relativity.

In Chapter 4, the method of İnönü-Wigner contractions is reviewed and then applied to the quaplectic group. Just as the Poincaré group contracts to the Galilean group in the limit $c \to \infty$, the quaplectic group contracts to various physically significant subgroups. The Hamilton group, which is the group of non-inertial classical mechanics, the Poincaré group and the Galilean group can all be obtained from the quaplectic group using this method. Much of the work in this chapter is original but unpublished, and has been done in close collaboration with Stephen Low. The contraction of the Poincaré group to the Galilean group occurs because the Poincaré symmetry is a deeper symmetry than Galilean symmetry; indeed, Galilean symmetry only manifests itself in the case $c \to \infty$, which is precisely the contraction limit. It is possible that the Poincaré symmetry is itself the contraction of a deeper symmetry group. This chapter suggests that the quaplectic group is a candidate for this deeper symmetry group.

Chapter 5 presents the branching rules and state labelling problem for the group $\mathcal{Q}(2) \supset \mathcal{E}(2)$ on 2 dimensional Euclidean space. Most of the material in this chapter is original and has been done in close collaboration with Peter Jarvis. This chapter explores the possibility of quaplectic symmetry being a higher-dimensional symmetry. If this is the case then standard particle classifications should be generalised under this symmetry. This is analogous to having a relativistic wave equation for some extended space-time symmetry group and determining the particle multiplets as $R \to \infty$.

Chapter 6 presents the Schrödinger-Robertson inequality as it applies with quaplectic symmetry. For the scalar case of quaplectic symmetry the Schrödinger-Robertson inequality becomes: $\det(\Sigma) \geq (\frac{1}{2}\hbar)^8$ where Σ is the covariance matrix. It is also shown that $\det(\Sigma)$ is invariant under quaplectic transformations, and the consequences of this are studied. This chapter is based on a paper [42] written in collaboration with my supervisor, Peter Jarvis.

Chapter 7 presents the worldline quantisation of a reciprocally invariant system, and is based on two papers. Each paper uses a different method to diagonalise the Hamiltonian. The first paper [35], which uses Lorentz covariant Fock generators (with a Lorentz invariant vacuum state), was written in collaboration with Jan Govaerts, Peter Jarvis and Stephen Low. The second paper [43] makes a different choice of Fock generators such that there are no negative norm states and a non-invariant Fock vacuum state. This paper was written in collaboration with Peter Jarvis.

Finally, the thesis is concluded and further work suggested in Chapter 8.

1.2 Max Born

Max Born is the biographical subject of Nancy Thorndike Greenspan's book *The End of the Certain World* [36]. Written in 2005, this is the first biography written on the life of Born. Most of the details of this section are taken from that book, or from [15], [5] or [1].

Max Born was born on the 11th of December 1882 in Breslau, Prussia (modern

day Wroclaw, Poland). He was tutored at home for a while before completing his secondary studies at the Kaiser Wilhelm Gymnasium in Breslau and matriculating to the University of Breslau. At the University he met Jakob Rosanes, a professor who taught Born the algebra of matrices.

With his interest in mathematics piqued, Born gravitated towards the University of Göttingen in 1904 to work on his PhD. At Göttingen he studied astronomy under Karl Schwarzchild and physics under Woldemar Voigt. He also met "the mandarins", Felix Klein, Hermann Minkowski and David Hilbert. He soon became the lecture scribe for Hilbert, who did not lecture from a text, and developed a close relationship with both Hilbert and Minkowski. He received his PhD on elasticity under Klein while being supervised by Carl Runge. This was in 1906, and he was then compulsorily conscripted into the German army before being discharged after a short service due to his asthma. He briefly attended Gonville and Caius College at Cambridge where he studied under Sir Joseph "J. J." Thomson, before returning to Breslau.

It was at Breslau in 1908 that he first encountered Albert Einstein's theory of special relativity. He wrote to Minkowski for some clarification on Einstein's paper. Minkowski quickly replied and invited Born to work as his assistant at Göttingen on this very subject. Unfortunately, Minkowski died soon after, in January 1909, due to a ruptured appendix. Born was able to complete and publish a reconstruction of some of Minkowski's work [22], and went on to do some lecturing at the University. It was at about this time that Born's primary interests shifted from mathematics to physics.

In 1912 he gave some lectures on relativity in Chicago and while there he assisted Albert Michelson with some experiments.

In 1913 he married Hedwig ("Hedi", née Ehrenberg) and also converted to Lutheranism, largely to mollify Hedi's converted Jewish parents. While he did not hide his Jewish roots, he later wrote that he did not regret his conversion. Max and Hedi had three children, including a daughter Irene; the mother of British-born Australian singer and actress, Olivia Newton-John. Max and Hedi's marriage was often turbulent; in particular, Hedi almost left Max at one stage for Gustav Herglotz.

In 1915 Born accepted a position at the University of Berlin as an assistant professor, assisting Max Planck. It was here that he first met Einstein, in 1916. Both had a love of physics and of music: Einstein being accomplished on the violin and Born on the piano. Both Born and Einstein have been described as "pacifists" by biographers, although Born would serve again in the German army during the First World War. The two would have a life-long friendship - a friendship that was turbulent at times and included many disagreements, both in physics and in their opinion of post-war Germany. Some of their letters were published by Born in a book [15], which includes commentary from Born.

Born was a patriotic German, a quality that remained throughout his life. He continued at the University of Berlin after his military service, and at the conclusion of the War he was appointed professor at the University of Frankfurt-on-Main, where his assistant was Otto Stern. He would then return to Göttingen in 1921 as a full professor and the head of his own department. His assistants in

this position included Wolfgang Pauli, Werner Heisenberg, Enrico Fermi, Eugene Wigner, Gerhard Herzberg and Maria Goeppert-Meyer - all of whom would go on to win Nobel prizes. He was also the PhD supervisor for other students such as Walter Heitler, Pascual Jordan and J. Robert Oppenheimer. His collaborators included Paul Dirac and Born's long-time friend James Franck.

In 1925, Heisenberg wrote his seminal paper, *Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen* (On a quantum-theoretical interpretation of kinematic and mechanical relations) [39]. This paper was published later that year. At the time he wrote it Heisenberg was unfamiliar with matrix algebra, so in effect he had to "reinvent the wheel" in writing this paper. Once Born had read the paper, he immediately recognised the algebra due to his earlier work with Rosanes at Breslau. Heisenberg's paper (an English translation of which can be found in [77]) contained the incredibly important and equally unrecognisable equation, in the context of emission spectral lines:

$$h = 4\pi m \sum_{\alpha=0}^{\infty} \{|a(n, n+\alpha)|^2 \omega(n, n+\alpha) - |a(n, n-\alpha)|^2 \omega(n, n-\alpha)\},$$

where the a's represent the probability amplitude of coordinates and the ω's represent the frequency of the spectral lines. There is little doubt that the equation is purely Heisenberg's invention, but the form in which we know it today is mostly due to Born. He identified one of the a's in each square as a position variable q, and the other combining with the frequency and the mass m to form a momentum variable p. This allowed him to rewrite the equation as follows:

$$pq - qp = \frac{h}{2\pi i}. \tag{1.1}$$

In fact this very equation is engraved on Born's tombstone. In addition, it was Born's intuition that the non-diagonal elements would commute ($[q_i, p_j] = 0$, for $i \neq j$) and this intuition was confirmed by Jordan at Born's direction.

The fact that Born did not share Heisenberg's Nobel prize in 1932 came as a surprise - at least to Born and to Heisenberg. Heisenberg wrote a letter to Born expressing his embarrassment over the omission. The matter was exacerbated in Born's mind since Dirac received the 1933 award with Erwin Schrödinger. Although Dirac was a worthy winner regardless, the award was ostensibly for work that Born himself published before Dirac. Born was helped with this work by Jordan - who by the time the award was presented in 1933 was a member of the Nazi party. Bernstein [5] speculates that the reason Born did not receive the Nobel prize was due to the involvement of Jordan. When Born was finally awarded the prize in 1954 for his work on the statistical interpretation of quantum mechanics, Born's previous work was not mentioned by the committee, supporting this hypothesis. For some insight into how Born felt about Jordan, Bernstein recounts that after the war Jordan wrote to Heisenberg and Born requesting a testimonial for a promotion he was seeking in Hamburg. Heisenberg provided him with such a testimonial, whereas Born wrote Jordan a letter listing the names of his family members who had been murdered by the Nazis.

In 1933 Born was forced by the Nazis to resign from his position at Göttingen and so he decided to emigrate. He returned to Cambridge in England where he taught for three years as Stokes Lecturer and collaborated with Leopold Infeld. He then went to the University of Edinburgh where he was appointed Tait Professor of Natural Philosophy in 1936. Before the war, Born endeavoured to secure positions in foreign universities for as many German Jewish physicists as possible. Most of those who could not leave died.

At Edinburgh he supervised Herbert S. Green and Klaus Fuchs, who both worked on reciprocity with Born (Fuchs would go on to work on the Manhattan Project and become infamous as a Soviet spy). Bernstein expressed skepticism over Born's idea of reciprocity. Perhaps, however, this will be his enduring legacy.

Max Born died on the 5th of January 1970 and was buried in Göttingen.

1.3 Notation

Quantities in four-dimensional space-time or momentum-energy will use indices from the Greek alphabet. These indices generally run from 0 to 3 but can instead run from 0 to n if needed. Space-time and momentum-energy four-vectors are denoted by x_μ and p_μ respectively, where:

$$\begin{aligned} x_0 &= ct, & p_0 &= e/c \\ x_1 &= x, & p_1 &= p_x \\ x_2 &= y, & p_2 &= p_y \\ x_3 &= z, & p_3 &= p_z, \end{aligned}$$

and where c is the speed of light; t is coordinate time; x, y and z are space coordinates; p_x, p_y and p_z are the corresponding momentum coordinates in their respective directions, and e is the energy coordinate. This differs from most notation which uses a capital E for energy. In most contexts in this thesis, capital letters have been used to indicate operators and lower case letters are therefore used for the eigenvalues of the operators or for coordinates. This creates the potential for confusion between the energy coordinate and Euler's constant; also denoted by e. However, the context should make it clear which e is used in each particular situation. The corresponding contravariant vectors are obtained by use of the raising metric. Throughout this thesis I have used the following Minkowski metric:

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.2}$$

Explicitly, the contravariant and covariant tensors are related by:

$$\begin{aligned} x^\mu &= \eta^{\mu\nu} x_\nu, & x_\mu &= \eta_{\mu\nu} x^\nu \\ p^\mu &= \eta^{\mu\nu} p_\nu, & p_\mu &= \eta_{\mu\nu} p^\nu, \end{aligned}$$

giving, for each component:

$$\begin{aligned} x^0 &= -x_0, & p^0 &= -p_0 \\ x^1 &= x_1, & p^1 &= p_1 \\ x^2 &= x_2, & p^2 &= p_2 \\ x^3 &= x_3, & p^3 &= p_3. \end{aligned}$$

Another useful metric is the *symplectic metric* ω. For a space-time-momentum-energy vector Z, defined (in one spatial dimension) by:

$$Z = \begin{pmatrix} t \\ x \\ p \\ e \end{pmatrix}, \tag{1.3}$$

the symplectic metric is given by:

$$\omega := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.4}$$

The *symplectic line element* is defined as:

$$dZ^T \omega dZ = dp \wedge dx - de \wedge dt, \tag{1.5}$$

where the superscript T refers to the transpose of the vector Z. The *symplectic transpose* of a matrix z is then given by $z^T\omega$. For example, if a is a general element of the translation algebra (see Section 2.5 as well as Section 2.6.3) in one time, space, momentum and energy dimension then this can be represented by matrices:

$$a = \begin{pmatrix} 0 & 0 & 0 & 0 & \tau \\ 0 & 0 & 0 & 0 & \xi \\ 0 & 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 0 & \eta \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where the τ, ξ, π and η refer to a translations in time, space, momentum and energy respectively. The 4×4 symplectic metric ω must be embedded in a 5×5 matrix by adding a fifth row and column of zeros. The symplectic transpose of a is then given by:

$$a^T\omega = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\eta & \pi & -\xi & \tau & 0 \end{pmatrix}.$$

The space-time-energy-momentum vector Z of Equation 1.3 will be referred to as "phase space" and it will be used throughout the thesis. If there are n spatial

dimensions, there will also be n momentum directions and the phase space vector will be modified to:

$$Z = \begin{pmatrix} t \\ x_i \\ p_i \\ e \end{pmatrix}. \tag{1.6}$$

A label will be used to indicate how many spatial (and therefore momentum) coordinates there are in a particular phase space, so that the vector Z in Equation 1.3 will be said to refer to "1-phase space" and the vector Z in Equation 1.6 will be said to refer to "n-phase space".

For both space-time and momentum-energy vectors (or indeed for any other $(1+n)$ or $(1+3)$ tensors), the 0 component will be called "timelike" and the remaining components called "spacelike". Quantities in three-dimensions will use indices taken from the middle of the Roman alphabet, for example i, j or k. These indices all run from 1 to 3. The metric for three-dimensional quantities (or n-dimensional quantities) is simply the identity matrix in three-dimensions (or n-dimensions).

The Einstein summation convention is used throughout this thesis, so that repeated indices are summed over.

1.3.1 Dimensions

The coordinates mentioned in Section 1.3 are all dimensional variables. That is, x might be measured in metres, t in seconds, p in kilograms metres per second and e in Joules. It is useful to follow Low's convention[1] and change to dimensionless variables using scales λ_i, $i = t, x, p, e$ as follows:

$$\lambda_t = \sqrt{\frac{\hbar}{bc}} \qquad \lambda_x = \sqrt{\frac{\hbar c}{b}} \tag{1.7}$$

$$\lambda_p = \sqrt{\frac{\hbar b}{c}} \qquad \lambda_e = \sqrt{\hbar bc}. \tag{1.8}$$

These scales depend on two fundamental constants: Planck's constant $\hbar$ and the speed of light c. They also depend on a postulated third constant b. This has units of force and was proposed by Low to be a maximum measure of force. Just as c is the maximum rate of change of position, b is the maximum rate of change of momentum. The role of this constant is discussed more in Section 3.1. The scales relate to one another as follows:

$$\begin{array}{ccccccc} \lambda_t & = & \frac{1}{c}\lambda_x & = & \frac{1}{b}\lambda_p & = & \frac{1}{bc}\lambda_e \\ \lambda_x & = & c\lambda_t & = & \frac{c}{b}\lambda_p & = & \frac{1}{b}\lambda_e \\ \lambda_p & = & b\lambda_t & = & \frac{b}{c}\lambda_x & = & \frac{1}{c}\lambda_e \\ \lambda_e & = & bc\lambda_t & = & b\lambda_x & = & c\lambda_p. \end{array}$$

[1] See, for example, [54]

These scales correspond to the Planck units via the identification:

$$G = \alpha_G \frac{c^4}{b},$$

where G is the gravitational constant. The value of the dimensionless constant α_G - and hence of b - must be determined by experiment. If, for example, it were determined that $\alpha_G = 1$, then the scales in Equations 1.7 and 1.8 would correspond exactly to the Planck units. But there is no reason to expect this to be true, and so the value of b remains to be determined.

The constants b, c and $\hbar$ can be related to the λ_i scales in various ways. These are most easily displayed by using the following reproduced diagram of Low's[2]:

$$\begin{array}{ccccc} \lambda_t & \leftarrow & c = \frac{\lambda_q}{\lambda_t} & \rightarrow & \lambda_q \\ \uparrow & \nwarrow & & \nearrow & \uparrow \\ b = \frac{\lambda_p}{\lambda_t} & & \hbar = \lambda_t \lambda_e = \lambda_q \lambda_p & & b = \frac{\lambda_e}{\lambda_q} \\ \downarrow & \swarrow & & \searrow & \downarrow \\ \lambda_p & \leftarrow & c = \frac{\lambda_e}{\lambda_p} & \rightarrow & \lambda_e \end{array}$$

For the rest of this thesis, all variables will be measured in terms of these scales, unless otherwise specified. Implicit measurement in these scales allows the identification of $c = b = \hbar = 1$ and so these fundamental constants can be excluded where it is not useful to explicitly include them. Conversely, it is sometimes useful to add the dimensions back into dimensionless variables. This can be done by sensibly dividing out the appropriate scales. For example, the line element of $1+1$-dimensional space-time in dimensionless variables is given by:

$$ds^2 = -dt^2 + dx^2. \tag{1.9}$$

This can be converted back to the dimensional version by the substitution:

$$ds \rightarrow \frac{d\tilde{s}}{\lambda_t} \qquad dt \rightarrow \frac{d\tilde{t}}{\lambda_t} \qquad dx \rightarrow \frac{d\tilde{x}}{\lambda_x}.$$

Equation 1.9 then becomes:

$$\frac{d\tilde{s}^2}{{\lambda_t}^2} = -\frac{d\tilde{t}^2}{{\lambda_t}^2} + \frac{d\tilde{x}^2}{{\lambda_x}^2}$$

or in units of time:

$$\Rightarrow d\tilde{s}^2 = -d\tilde{t}^2 + \frac{1}{c^2} d\tilde{x}^2.$$

It is sometimes useful to combine the space-time and momentum-energy components into a single vector, as in the 1-phase space vector Z in Equation 1.3. This extended space will be labelled by upper-case Roman characters M, N and so on, with $M, N = 0, 1, ..., 2n+2$ where n is the number of spatial coordinates. When this is done, the 0 component will refer to time and the next n components will refer to the spatial coordinates. The n components from $n+1$ to $2n+1$ will refer to the momentum coordinates, and the $2n+2$ component will refer to the energy coordinate.

[2]See, for example, [55]

1.4 Born-Green Reciprocity

In 1938, Max Born [6] suggested a simple yet elegant theory to link quantum theory and relativity. It was subsequently further developed by Born and Fuchs in [11, 12]; by Fuchs in [31]; by Landé in [47, 48]; by Born and Green in [13] and with Cheng in [10]; by Born and Rodriguez in [14] and by Born on his own in [7, 8, 9]. The theory of Born's 1938 paper [6] leads to as Born says:

> "a kind of granular or lattice structure of space without introducing such a strange assumption *a priori*."

This idea of Born's will be further taken up in Chapter 6. The paper starts with the motion of a free particle, which he represented by a plane wave:

$$\psi(x) = Ae^{\left[\frac{i}{\hbar}p_\mu x^\mu\right]}.$$

Any square-integrable wavefunction ψ in the x-space can be transformed to a wavefunction ϕ in the p-space by a Fourier transform:

$$\phi(p) = \int \psi(x)e^{\left[\frac{i}{\hbar}p_\mu x^\mu\right]}dx.$$

Similarly, a wavefunction ϕ in the p-space can be transformed back to the x-space via a Fourier transform. Additionally, the representation of the momentum operators P^μ in x-space is $-i\hbar\frac{\partial}{\partial x^\mu}$ whereas position operators X^μ in the p-space representation is $i\hbar\frac{\partial}{\partial p^\mu}$. These two similarities between space-time and momentum-energy, inherent to the fundamental laws of quantum mechanics, suggested to Born a theory he called the "principle of reciprocity". He chose this word to describe the theory since, in his own words [6]:

> "The word 'reciprocity' is chosen because it is already generally used in the lattice theory of crystals where the motion of the particle is described in the p-space with help of the 'reciprocal lattice'."

The principle can be succinctly stated thus: the laws of physics must be invariant under the following discrete transformation:

$$x^\mu \to p^\mu, \qquad p^\mu \to -x^\mu. \tag{1.10}$$

Clearly this transformation does not keep invariant many of the laws of physics as we know them. If the principle is to be true, then it must also be true that the laws of physics must be able to be extended to a reciprocally invariant law such that the extended law converges to the currently known law in the non-reciprocal limit. This must be true in the same way that Galilean mechanics is the inertial non-relativistic limit of special relativity. The problem of extending Newton's law of gravitation to a reciprocally invariant law has begun to be studied by Lashmar and Delbourgo [25, 50].

The transformation 1.10 clearly leaves invariant the afore-mentioned aspects of quantum mechanics. Any function (or functional) S of the form:

$$S(X, P) = S(P, -X)$$

is said to be *reciprocally invariant*, and eigenfunctions (or functionals) F which satisfy:

$$\langle x|\, S(X,P)\,|F\rangle = s\,\langle x|F\rangle$$

are said to be *self-reciprocal*. Born argued that functions F which satisfy this requirement are their own Fourier transforms[3]:

$$F(x) = \frac{1}{\sqrt{2\pi}}\int F(p)e^{-ipx}dp, \tag{1.11}$$

and he used this property as an alternative definition of self-reciprocity.

Born reciprocity is immediately recognised as being apparent in Hamilton's equations:

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu}, \qquad \dot{p}_\mu = -\frac{\partial H}{\partial x^\mu}.$$

An immediate consequence of Born's reasoning is the capability to break up Plank's constant into two factors:

$$\hbar = ab, \tag{1.12}$$

where a was some "minimum" length and b a momentum. Taking the value of $\hbar$ as unity and writing coordinates x^μ in units of a and momenta p^μ in units of b enabled Born to use dimensionless units. In fact, this length corresponds to the length scale λ_x defined in Equation 1.7 and the momentum corresponds to the momentum scale λ_p defined in Equation 1.8 (Low's constant b used in these scales should not be confused with Born's b defined in Equation 1.12).

The next application was to use this principle to identify the rest masses of fundamental particles. To do this, Born posited that particles with integral spin are supposed to have a wave function $|\psi\rangle$ which satisfies a wave equation of the form:

$$P^\mu P_\mu\,|\psi\rangle = \kappa^2\,|\psi\rangle\,, \tag{1.13}$$

[3]Functions satisfying $F(x) = \int F(p)e^{-ipx}đp$ include Gaussians; however, without specifying them further, a heuristic argument can be introduced for examining eigenstates of reciprocal invariant operators acting on them. Firstly note that, from basic Fourier properties:

$$xF(x) = \int -i\frac{\partial}{\partial p}F(p)e^{-ipx}đp, \qquad i\frac{\partial}{\partial x}F(x) = \int pF(p)e^{-ipx}đp$$

where $đp := dp/2\pi$. Then applying $S(x,p) \to S(x,-i\partial_x)$ to obtain such an F:

$$S(x,-i\frac{\partial}{\partial x})F(x) = \int S(-i\frac{\partial}{\partial p},p)F(p)e^{-ipx}đp = \int S(p,-i\frac{\partial}{\partial p})F(p)e^{-ipx}đp,$$

so the space of functions F is closed under the action of self-reciprocal operators. It then makes sense to ask for functions which diagonalise them, i.e. to investigate the spectrum and eigenfunctions of such S. See [13]. However, this does not prove that self-reciprocal functions are their own Fourier transforms, and so it is unclear as to what Born had in mind here.

where the eigenvalue κ is proportional to the rest mass of a fundamental particle. To obtain other eigenvalues (and hence other theoretical particles), note that this equation could equally well be written as:

$$\langle x| F(P) |\psi\rangle = 0,$$

where $F(P) = P^\mu P_\mu - \kappa^2$. Alternatively, $F(P^\mu)$ could be defined as:

$$F(P) = F_1(P^\mu)(P^\mu P_\mu - \kappa^2),$$

for some other κ, say κ_1, where $F_1(P) = 0$ has no roots. On the other hand, $F_1(P)$ may have roots and it could therefore be defined as:

$$F_1(P) = F_2(P)(P^\mu P_\mu - {\kappa_2}^2),$$

and so on *ad infinitum*. It can be seen therefore that there are possibly an infinite number of distinct eigenvalues κ_i, $i = 1, 2, ...$ that can be determined by a suitable choice of F in Equation 1.13 and these eigenvalues all correspond to an actual fundamental particle.

For any eigenvalue κ_i the actual rest mass μ_i is then:

$$\mu_i = \frac{\hbar}{ac}\kappa_i, \tag{1.14}$$

where a is the length constant of Equation 1.12. If the eigenvalues κ can be determined, then it remains to identify each theoretical rest mass κ with the correct actual rest mass μ. If this can be done for even one particle, then the constant a can be determined and then each of the rest of the eigenvalues κ_i can easily be identified with its corresponding particle. Determining these eigenvalues κ_i and the corresponding rest masses of actual particles was the purpose of Born's 1949 paper [7]. He also proposed [7] using the same method to find the theoretical rest masses of spin-half particles by taking the "square root" of Equation 1.13 and replacing $P^\mu P_\mu$ with $\gamma^\mu P_\mu$, where $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ and γ^μ are the Dirac matrices.

To find these theoretical mass values, Born used the simplest reciprocally invariant function $S(X, P)$ he could think of: the metric operator in 3-phase space[4]:

$$S(X, P) = X^\mu X_\mu + P^\mu P_\mu.$$

He then solved the eigenvalue equation:

$$\begin{aligned} &\langle p| X^\mu X_\mu + P^\mu P_\mu |F\rangle = s |F\rangle \\ \Rightarrow \qquad &(-\frac{\partial^2}{\partial p^\mu \partial p_\mu} + p^\mu p_\mu - s)F(p) = 0, \end{aligned} \tag{1.15}$$

by factorising

$$F = F_k(M)Y_k(\theta, \phi, \omega), \qquad M := P^\mu P_\mu = p^2,$$

[4]See Section 1.3 for the definition of 3-phase space.

n	k	s	κ				
1	0	4					
2	1	6	0				
	0	8	0,	1,41[a]			
3	2	8	0				
	1	10	0,	1.73			
	0	12	0,	2.13,	2.17[b]		
4	3	10	0				
	2	12	0,	2.0			
	1	14	0,	1.41,	2.45		
	0	16	0,	0.97,	1.82,	2.78	
5	4	12	0				
	3	14	0,	2.24			
	2	16	0,	1.66,	2.70		
	1	18	0,	1.20,	2.07,	3.06	
	0	20	0,	0.86,	1.60,	2.39,	3.31

Table 1.1: Calculated Theoretical Rest Masses

where Y_k is a four-dimensional harmonic. With boundary conditions $F_k \to 0$ as $M \to \infty$, the solution is:

$$s = 2(2n - k); \qquad F_k = M^{\frac{k}{2}} e^{-\frac{M}{2}} L_n^{(k+1)}(M); \tag{1.16}$$

a solution involving derivatives of Laguerre polynomials $L_n^{k+1}(M)$. The index $k = 0, 1, ...$ refers to the order of the derivative of the Laguerre polynomial of order $n \geq k + 1$. This leads to an infinite spectrum of rest masses of theoretical particles determined by the roots of the equation:

$$\kappa^k L_n^{(k+1)}(\kappa^2) = 0.$$

For particles of integral spin, Born was able to calculate the theoretical rest mass values for κ seen in Table 1.4:

At the time Born wrote this paper the quark model had not yet been proposed and an infinite number of particles was considered to be a possibility. Born, together with Antonio Rodriguez, attempted to fix the ratio of theoretical rest mass to actual rest mass seen in Equation 1.14 by proposing as an approximation that the length constant a be the classical radius of the electron [14]. This initially yielded some encouraging results, with the two able to identify the "first" value of κ - labelled by the superscript a in Table 1.4 - of $\kappa = 1.41$ as being the mass of the "most stable meson". By the "most stable meson", Born and Rogriguez were most likely speaking of the muon. This has a mass of 105.66MeV/c^2, which is roughly 207 times the mass of the electron at 0.51MeV/c^2. They were also able to identify the "second" value of κ - labelled by b in Table 1.4 - of $\kappa = 2.17$ as

being the mass of one of the pi-mesons with a mass roughly 280 times the mass of the electron. Of course, the muon has since been discovered to not be a meson at all. Additionally, it has half-integral spin (which was not known at the time) and so should not be predicted by this model. Finally, the ratios of actual masses do not match up well with the theoretical ratios, now that the actual masses are known with more precision.

Though the identities Born made were approximate only, Born thought that it would just be a matter of time for the experimentalists to catch up to the theory, and so he was not worried by this. In fact, Born was so confident in the eventual success of this theory that he boasted to Einstein [15]:

> "Green and I have developed a theory for elementary particles, and I am convinced that it is correct, though I express myself a little more cautiously in the literature. You will not believe in it, [however]... the masses of the known mesons are calculated correctly. No offence!"

Einstein replied:

> "our respective hobby-horses have irretrievably run off in different directions – yours, however, enjoys far greater popularity as a result of its remarkable practical successes, while mine, on the other hand, smacks of quixotism, and even I myself cannot adhere to it with absolute confidence. But at least mine does not represent a blind-man's bluff with the idea of reality. My whole instinct rebels against it irresistibly."

The experimentalists have since confirmed Einstein's intuition, as this model fails to identify theoretical rest masses with actual fundamental particles. There is also no reason today to suspect that there are an infinite number of mesons.

Both of these problems make the theory of the 1949 paper [8] fundamentally and irretrievably flawed in trying to predict the meson mass spectrum from Equation 1.13. But does this invalidate the entire thesis of the reciprocity of position and momentum? In other words, is Born's hypothesised principle that the laws of physics are invariant under the transformation of Equation 1.10 necessarily invalid because of the failure of the theory to predict the mass spectrum of fundamental particles? In the opinion of the author, the answer to these questions is no. There is another assumption Born makes that is readily seen to be invalid: it is the assumption that particles should satisfy a wave function in the form of Equation 1.13 in the first place. This assumption seems entirely arbitrary and without basis or merit. On the contrary, a much more satisfying and elegant theory is that developed from Wigner [81] onwards, that the intrinsic properties of fundamental particles such as spin and mass can be derived from field equations which are the eigenvalue equations of the Casimir operators of the Lie algebra of symmetry groups. Mass arises as an eigenvalue of the Klein-Gordon equation and the Dirac equation which are both field equations derived from the Poincaré group (see Section 2.8).

If Born reciprocity is therefore to be retrieved, a suitable task is to find if it is a symmetry of a group, and then to find the Casimir operators and hence the field equations of that group. If the eigenvalues thus derived correspond to observed

phenomena, then Born's reciprocity theory will be validated. This task has been begun by Stephen Low (cf. Section 3.1).

Another avenue of investigation has been pursued by Caianiello [17]. In Newtonian mechanics, a metric could be proposed such that: $S^2 = t^2 + \alpha x^2$. The expectation value is given by: $\langle S^2 \rangle = \overline{t}^2 + \alpha \overline{x}^2 + {\sigma_t}^2 + \alpha {\sigma_x}^2$, for standard deviations in time and position given by σ_t and σ_x respectively. To make the move from Newtonian mechanics to special relativistic mechanics requires the imposition of the condition: $\frac{\sigma_x}{\sigma_t} \leq c$. The choice $\alpha = -1/c^2$ gives:

$$\langle S^2 \rangle = \overline{t}^2 - \frac{1}{c^2}\overline{x}^2 + {\sigma_t}^2 - \frac{1}{c^2}{\sigma_x}^2$$

Caianiello then proposes a phase-space metric similar to Low's (cf. Chapter 3 and Equation 3.1)[5]:

$$S^2 = t^2 - \frac{1}{c^2}x^2 + \frac{\alpha}{c^2}\left[\frac{1}{c^2}e^2 - p^2\right]. \tag{1.17}$$

Caianiello argues that Equation 1.17 is not measurable unless the standard deviations σ_x and σ_p are suitably related through the quantity α, as were σ_x and σ_t above in the case of special relativity. Heisenberg's uncertainty principle (in one dimension) demands:

$$\sigma_x \sigma_p \geq \hbar.$$

Caianiello then makes the choice:

$$\alpha = \frac{{\sigma_x}^2}{{\sigma_p}^2}\alpha_0 \varepsilon, \tag{1.18}$$

where $\alpha_0 > 0$ and $\varepsilon = -1$ or $+1$. Then, by replacing σ_x with λ and σ_p with μc and setting $\hbar = \lambda \mu c$, Equation 1.17 becomes:

$$S^2 = t^2 - \frac{1}{c^2}x^2 + \frac{\hbar^2 \varepsilon}{\mu^4 c^6}\left[\frac{1}{c^2}e^2 - p^2\right].$$

From the Minkowski metric $c^2 ds^2 = c^2 dt^2 - dx^2$, one can obtain the result that no velocity can exceed the speed of light by requiring $c^2 ds^2 \geq 0$. A similar requirement for Equation 1.18 means that:

$$c^2 dS^2 = c^2 dt^2 - dx^2 + \frac{\hbar^2 \varepsilon}{\mu^4 c^4}\left[\frac{1}{c^2}de^2 - dp^2\right] \geq 0$$

$$\Rightarrow c^2 - v^2 + \varepsilon \frac{\hbar^2}{\mu^4 c^4}\left[\frac{1}{c^2}\left(\frac{de}{dt}\right)^2 - \left(\frac{dp}{dt}\right)^2\right] \geq 0$$

[5]This equation is reproduced in the same form as in [17], except that the energy e is in lower case so as to preserve consistency of notation.

Using the standard formulae of special relativity but without going into rigorous detail, Caianiello argues that this then becomes:

$$(c^2 - v^2)\left[1 - \varepsilon\frac{\hbar^2}{\mu^4 c^4}\frac{{m_0}^2 a^2 c^4}{(c^2 - v^2)^2}\right] \geq 0,$$

where a is the acceleration in one dimension - however, the argument remains similar for three dimensions. The choice $\varepsilon = -1$ does not yield a maximal acceleration. On the other hand, the choice $\varepsilon = +1$ yields a maximal acceleration $a_{\max}$ of:

$$a = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2} \leq a_{\max};$$
$$a_{\max} = \frac{\mu c^2}{m_0 \lambda}.$$

Caianiello's idea of a maximal acceleration is somewhat similar to Low's idea of a maximal force (cf. Chapter 3). Caianiello's idea is further examined by several others, including Caianiello *et al* [18], Feoli [29], Landolfi [49], and Castro [19, 21, 20].

Chapter 2

Semidirect Products & Group Representations

This chapter reviews some important definitions and results in group theory and the representations of those groups. Section 2.1 is a brief summary of some basic definitions in group theory. These definitions are provided for handy reference in later sections and chapters. Section 2.2 reviews a truly remarkable result of group representation theory, due to Wigner [81] and Bargmann [3]: quantum mechanical states are described by rays in a Hilbert space, and these rays can be represented by projective unitary representations of elements of a relevant symmetry group (usually taken to be the Lorentz/Poincaré group or the Euclidean/Galilean group). Projective unitary representations are not usually the easiest tools to work with, but they are isomorphic to unitary representations of a central extension of the symmetry group. There are two possibilities for centrally extending a group: algebraically and topologically; a topological central extension usually being called a "covering group" of the symmetry group. Projective representations are discussed at the beginning of Section 2.2. Algebraic central extensions are then discussed in Section 2.2.1 and topological central extensions are discussed in Section 2.2.2.

Section 2.3 introduces the concept of a semidirect product group. This is a product between two groups, a "homogeneous" group and a "normal" group. The important definitions of "little groups" and "stabiliser groups" are discussed in Section 2.3.1. The case of the normal subgroup being abelian is then discussed in Section 2.3.2. Section 2.4 then describes the Mackey induction method. This is a method of constructing the representations of a semidirect product group from the representations of its homogeneous and normal subgroups. The remainder of Chapter 2 is devoted to looking at several examples of symmetry groups and the representations of those groups. Section 2.5 looks at the translation group, which is an abelian group. Section 2.6 looks at the Weyl-Heisenberg group, which is a semidirect product of two translation groups. It is the symmetry group of the canonical commutation relations. Since it is a semidirect product group the representations can be constructed using the Mackey method of induced representations, and this is done in Section 2.6.2. Section 2.7 looks at the Lorentz group, which is the symmetry group of special relativity. The semidirect product of the Lorentz group and a translation group, involving translations in space and

time, is the Poincaré group[1]. The Poincaré group is studied in Section 2.8 and the representations for two cases - the timelike case and the massless case - are constructed using the Mackey method of induced representations in Section 2.8.2. The eigenvalue equations for the Casimir operators of symmetry groups define the field equations for those groups, and so Section 2.8.4 looks at the Casimir operators of the Poincaré group. The Klein-Gordon equation is derived as an example in Section 2.8.4.3. Other important equations in quantum mechanics such as the Dirac equation, Maxwell's equations and the Weyl equation can be derived in a similar fashion from the Poincaré group (see [26] for derivations of these).

2.1 Some Representation Theory Definitions

In this section, some standard representation theory concepts are defined. In Section 2.1.1, various morphisms (mappings) are defined, including the important homomorphism and isomorphism. In Section 2.1.2, the concept of a representation of a Lie group is introduced. In Section 2.1.3, the concept of a Lie algebra is introduced, and several standard results are defined. In particular, the reason for the existence of a factor of i in the canonical commutation relations is discussed in Section 2.1.3.1. Section 2.1.4 defines Schur's lemma, and Section 2.1.5 defines the concept of a Casimir operator. These are the basics of Lie group theory, and will be used throughout the thesis.

2.1.1 Morphisms

A group *homomorphism* is a mapping ρ from a group $\mathcal{G}$ to another group $\mathcal{H}$ that preserves the group product. It will be taken to be a function of elements of $\mathcal{G}$ with image in $\mathcal{H}$:

$$\rho : \mathcal{G} \to \mathcal{H};\ \rho(g) \in \mathcal{H} \text{ for } g \in \mathcal{G}.$$

Then for elements $g_1, g_2 \in \mathcal{G}$:

$$\rho(g_1)\rho(g_2) = \rho(g_1 g_2). \tag{2.1}$$

An *isomorphism* of a group $\mathcal{G}$ is a particular group homomorphism ρ, where ρ and its inverse ρ^{-1} are bijective. If there exists an isomorphism from $\mathcal{G}$ to another group $\mathcal{H}$, then these two groups are said to be *isomorphic* to one another. This will be denoted by:

$$\mathcal{G} \cong \mathcal{H}.$$

Additionally, a homomorphism that maps $\mathcal{G}$ to itself is called an *endomorphism*, and an isomorphism that maps $\mathcal{G}$ to itself is called an *automorphism*. In other words, an automorphism is a homomorphism that, along with its inverse, is a bijective endomorphism.

[1]Technically, this group is actually the inhomogeneous Lorentz group and the Poincaré group is a central extension of the inhomogeneous Lorentz group. This is explained further in Section 2.8

2.1.2 Lie Group Representations

A *representation* of a group $\mathcal{G}$ over a vector space V of dimension n is a homomorphism ρ that maps elements of $\mathcal{G}$ to the group of all automorphisms of V; $Aut(V)$. In other words, it is a map to $\mathcal{GL}(n)$; the group of all invertible operators of dimension n:

$$\rho : \mathcal{G} \to \mathcal{GL}(n).$$

This group is called the *General Linear Group*. If n is finite, then $\mathcal{GL}(n)$ is isomorphic to the group of $n \times n$ non-singular matrices. Indeed, $\mathcal{GL}(n)$ is virtually interchangeable with the group of $n \times n$ non-singular matrices, so much so that the group of all $n \times n$ non-singular matrices is usually called $\mathcal{GL}(n)$. A *matrix representation* is therefore a representation where the operators $\rho(g)$ are explicitly $n \times n$ non-singular matrices. Sometimes this is called a *matrix realisation* of the group.

For matrix representations, the group product becomes the matrix product, the inverse is given by the matrix inverse and the identity is the $n \times n$ unit matrix, which will be denoted I_n or $I_{n\times n}$.

2.1.3 Lie Algebra

A Lie algebra $\mathfrak{g}$ consists of a vector space V over a field F, together with a binary operation called the Lie bracket, denoted by $[,]$:

$$[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

For elements x, y and z in $\mathfrak{g}$, the Lie bracket is bilinear:

$$[ax + by, z] = a\,[x, z] + b\,[y, z]\,, \qquad [x, ay + bz] = a\,[x, y] + b\,[x, z]$$

where a and b are in F. The Lie bracket is skew-symmetric:

$$[x, y] = -\,[y, x]\,,$$

and it satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

A Lie algebra *homomorphism* ρ' is a mapping from $\mathfrak{g}$ to another Lie algebra $\mathfrak{h}$:

$$\rho' : \mathfrak{g} \to \mathfrak{h}, \qquad \rho'([x, y]) = [\rho'(x), \rho'(y)]\,.$$

A Lie algebra *isomorphism* is a Lie algebra homomorphism that is bijective, as is its inverse. Similarly to group isomorphisms, if there exists an isomorphism between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, then the Lie algebras are said to be isomorphic to each other. This is denoted by:

$$\mathfrak{g} \cong \mathfrak{h}.$$

Additionally, the concepts of endomorphisms and automorphisms are defined similarly to their Lie group counterparts (see Section 2.1.1).

For a Lie algebra $\mathfrak{g}$ on a vector space V, a Lie algebra *representation* is a homomorphism ρ' that takes $\mathfrak{g}$ to the set of all endomorphisms of V; $End(V)$. This is denoted by $\mathfrak{gl}(n)$ and called the *General Linear Lie Algebra*. This is isomorphic to the set of all real $n \times n$ matrices. This set differs from the General Linear Group in that the matrices no longer need to be non-singular; the matrices no longer need to be invertible. The Lie bracket of $\mathfrak{gl}(n)$ is given by the commutator:

$$[x, y] = xy - yx,$$

for $x, y \in \mathfrak{g}$.

Every Lie group has a corresponding Lie algebra. For a connected group $\mathcal{G}$, consider a parametrization of the group elements by a finite set of real parameters α^i. The number of parameters equals the dimension of the group. A general group element g can be written as a function of these parameters, $g = g(\alpha^i)$. By convention, choose:

$$g(0, 0, ..., 0) = e_g;$$

the identity element is such that the parameters α^i are all zero for that element. As all of the group elements can be expressed as a function g of the varying parameters, the g will be dropped in the notation, so that a general group element will be denoted by $\alpha \equiv \alpha^i = (\alpha^1, \alpha^2, ..., \alpha^r)$ where there are r parameters in this case. A second general group element will be written as $\beta \equiv \beta^i$.

Now consider a matrix representation of the group, $\rho(\alpha) \equiv \rho(\alpha^i)$, $\rho : \mathcal{G} \to \mathcal{GL}(n)$, with $i = 1, 2, ..., n$. This representation has first and second partial derivatives evaluated at the identity:

$$\rho'_i(e_g) = \left.\frac{\partial \rho(\alpha)}{\partial \alpha^i}\right|_{e_g}, \qquad \rho''_{ij} = \rho''_{ji} = \left.\frac{\partial^2 \rho(\alpha)}{\partial \alpha^i \partial \alpha^j}\right|_{e_g}.$$

Then, the representation of a general group element can be expanded in a Taylor series centred at the identity:

$$\rho(\alpha) = \rho(e_g) + X_i \alpha^i + \tfrac{1}{2} X_{ij} \alpha^i \alpha^j + ... \tag{2.2}$$

for small α^i and coefficients $X_i \equiv \rho'_i(e_g)$ and $X_{ij} \equiv \rho''_{ij}(e_g)$. The representation of the identity, $\rho(e_g)$, will be denoted by I_n - the $n \times n$ identity matrix. The multiplication of the representation of two group elements is given by:

$$\rho(\alpha)\rho(\beta) = \rho(f(\alpha, \beta)), \tag{2.3}$$

where $f(\alpha, \beta)$ is some function of the parameters. For Equation 2.3 to be a group multiplication law, the function $f(\alpha, \beta)$ must obey the following conditions:

$$f^i(0, \alpha) = f^i(\alpha, 0) = \alpha^i, \qquad f^i(0, 0) = 0. \tag{2.4}$$

Consider a power series expansion of this function centred at zero. For small α, β,

$$f^i(\alpha,\beta) = \alpha^i + \beta^i + f^i{}_{jk}\alpha^j\beta^k + \mathcal{O}(\alpha^2) + \mathcal{O}(\beta^2) + ... \tag{2.5}$$

for coefficients $f^i{}_{jk}$. Inserting $\beta = 0$ one can see immediately that the $\mathcal{O}(\alpha^2)$ terms (and likewise the $\mathcal{O}(\beta^2)$ terms) must vanish in order for Equation 2.4 to hold. From Equation 2.2, the left-hand-side of Equation 2.3 becomes:

$$\begin{aligned} LHS &= [I_n + X_i\alpha^i + \frac{1}{2}X_{ij}\alpha^i\alpha^j + ...][I_n + X_i\beta^i + \frac{1}{2}X_{ij}\beta^i\beta^j + ...] \\ &= I_n + X_i\alpha^i + X_i\beta^i + X_i\alpha^i X_j\beta^j + \frac{1}{2}X_{ij}\alpha^i\alpha^j + \frac{1}{2}X_{ij}\beta^i\beta^j + ... \end{aligned} \tag{2.6}$$

the right-hand-side of Equation 2.2 becomes:

$$\begin{aligned} RHS &= \rho(\alpha^i + \beta^i + f^i{}_{jk}\alpha^j\beta^k + ...) \\ &= I_n + X_i(\alpha^i + \beta^i + f^i{}_{jk}\alpha^j\beta^k + ...) + \frac{1}{2}X_{ij}(\alpha^i + \beta^i + ...)(\alpha^j + \beta^j + ...) \\ &= I_n + X_i(\alpha^i + \beta^i + f^i{}_{jk}\alpha^j\beta^k + ...) + \frac{1}{2}X_{ij}(\alpha^i\alpha^j + \beta^i\beta^j + 2\alpha^i\beta^j + ...). \end{aligned} \tag{2.7}$$

Subtracting terms common to both sides leaves the condition:

$$\begin{aligned} & X_iX_j\alpha^i\beta^j = X_kf^k{}_{ij}\alpha^i\beta^j + X_{ij}\alpha^i\beta^j \\ \Rightarrow \quad & X_{ij} = X_iX_j - X_kf^k{}_{ij} \end{aligned}$$

However, the second partial derivative is symmetric in i and j: $X_{ij} = X_{ji}$. Therefore:

$$\begin{aligned} & X_jX_i - X_kf^k{}_{ji} = X_iX_j - X_kf^k{}_{ij} \\ \Rightarrow \quad & X_iX_j - X_jX_i = X_if^k{}_{ij} - X_kf^k{}_{ji} \\ \Rightarrow \quad & [X_i, X_j] = C^k{}_{ij}X_k \end{aligned} \tag{2.8}$$

where $C^k{}_{ij} := f^k{}_{ij} - f^k{}_{ji}$ are a set of constants known as *structure constants*. Equation 2.8 defines a set of commutation relations that define a representation of a Lie algebra. This shows that a Lie algebra representation arises naturally from a Lie group representation, with the Lie algebra's vector space being the tangent space at the identity of the vector space of the Lie group. The set of X_i for $i = 1, 2, ..., n$ is a set of n generators of the representation of the Lie algebra.

For any connected Lie group $\mathcal{G}$, the identity component of $\mathcal{G}$, denoted by $\mathcal{G}_0$, is a (normal) subgroup of $\mathcal{G}$. The Lie group $\mathcal{G}_0$ is generated from a Lie algebra $\mathfrak{g}$ via the exponential map:

$$\mathfrak{g} \to \mathcal{G}_0 : e^x = g,$$

for $x \in \mathfrak{g}$ and $g \in \mathcal{G}_0$. The representations of the whole Lie group can then be derived from the Lie algebra representation by solving an extension problem. This

is discussed in Sections 2.2.1 and 2.2.2. As for the exponential map, if the group $\mathcal{G}_0$ in question is a matrix Lie group, then the exponential map is given by the usual Maclaurin series expansion:

$$e^x = \sum_{i=0}^{\infty} \frac{1}{i!} x^i.$$

An obvious example of this is for the group of all non-zero real numbers under multiplication. The corresponding Lie algebra is the group of all real numbers under addition. For groups other than matrix Lie groups, there is a more technical definition for the exponential map but the preceding definition suffices for the groups studied in this present work.

All matrix Lie groups are a subgroup of the *General Linear group*, $\mathcal{GL}(n)$. The associated Lie algebra representation, denoted by $\mathfrak{gl}(n)$, is the set of all $n \times n$ matrices, with Lie bracket given by the commutator of these matrices.

2.1.3.1 Hermitian Lie Algebra Representations

If the Lie group representations are unitary, then:

$$\rho(\theta)\rho^\dagger(\theta) = I.$$

Using the exponential map one finds that this requires a condition on the corresponding Lie algebra representation:

$$\begin{aligned} & e^{\rho'(x)} e^{\rho'(x)^\dagger} = I \\ \Rightarrow \quad & \rho'(x) + \rho'(x)^\dagger = 0 \\ \Rightarrow \quad & \rho'(x) = -\rho'(x)^\dagger. \end{aligned}$$

This means that the Lie algebra representations associated in a natural manner to the unitary representations of Lie groups are always skew-Hermitian. These skew-Hermitian representations can easily be replaced by equivalent Hermitian representations by multiplying $\rho'(x)$ by a factor of $i = \sqrt{-1}$:

$$\tilde{\rho}'(x) = i\rho'(x)$$

since

$$\begin{aligned} \tilde{\rho}'(x)^\dagger &= i\rho'(x)^\dagger \\ &= i(-i\tilde{\rho}'(x)) \\ &= \tilde{\rho}'(x). \end{aligned}$$

This is the reason for the existence of the i in the "canonical commutation relation" of position and momentum operators X and P (see Section 2.6):

$$[X, P] = iI.$$

For the rest of this thesis, even the representations of the generators corresponding to non-unitary group representations will usually be multiplied by i. The resulting Lie algebra will not therefore involve Hermitian generators necessarily, however the commutation relations thus obtained will be valid as well for the actual Hermitian representations of the generators.

2.1.4 Schur's Lemma

Let ρ be an irreducible representation of a group $\mathcal{G}$ with elements $g \in \mathcal{G}$ acting on a vector space V. Then if there exists an operator A that commutes with all of the representations $\rho(g)$, *Schur's lemma* (see, for example, [76]) states that A must be proportional to the representation of the identity element e_g of $\mathcal{G}$:

$$A = \lambda e_g,$$

with λ some coefficient of proportionality. If ρ is an n-dimensional matrix representation, then $A = \lambda I_n$. In other words, A is diagonal. Conversely, if the only operators that commute with all representations ρ of a group $\mathcal{G}$ are proportional to the representation of the identity, then ρ is an irreducible representation.

A consequence of Schur's lemma is the fact that all irreducible representations of abelian groups must be one-dimensional [76].

2.1.5 Casimir Operators

Casimir operators are combinations of the Lie algebra generators that commute with all of the generators of the Lie algebra. An irreducible representation of a Casimir operator is therefore proportional to the identity operator due to Schur's Lemma (see Section 2.1.4). The Casimir operator is an element of the centre of the universal enveloping algebra of a Lie algebra[2]. If the Lie Algebra generators are Hermitian then the eigenvalue equations of the corresponding Casimir operators give rise to the field equations relating to the underlying group symmetry. In the context of the Poincaré group, the Klein-Gordon Equation, the Dirac Equation, Maxwell's Equations and the Weyl Equation can all be derived using this method. The Klein-Gordon equation is derived in Section 2.8.4.3 as an example. For derivations of the other equations, see [26]. If the unitary irreducible representations of the group can be determined, then Hermitian irreducible representations of the Lie algebra generators can be calculated by differentiating the group representations at the identity (and multiplying the resulting representations by i - see Section 2.1.3.1). Combinations of these Lie algebra generators guarantees a Casimir operator that is Hermitian and irreducible.

2.2 Rays and Projective Representations

In the context of quantum mechanics, elements g of a relevant symmetry group $\mathcal{G}$ transform rays, $\mathcal{R}$:

$$g : \mathcal{R} \to \mathcal{R}'.$$

The ray $\mathcal{R}$ consists of vectors Ψ_r and the ray $\mathcal{R}'$ consists of vectors Ψ'_r. Two vectors in the same ray can differ only by a phase:

$$\Psi_1 = e^{i\phi(1,2)}\Psi_2, \qquad \Psi_1,\ \Psi_2 \in \mathcal{R},$$

[2]While the universal enveloping algebra is not defined here, it suffices to say that this means the Casimir operators are independent of any representation of a Lie algebra

where $\phi(1,2) \in \mathbb{R}$ depends in some way on Ψ_1 and Ψ_2.

Associated with any symmetry transformations g are operators U which act on the Hilbert space of physical states, and these operators must be either unitary and linear or antiunitary and antilinear. This result was proven by Wigner in [80], and there is a more complete proof in [78]. The product of two anti-unitary operators is unitary, which means that the representation of the identity of a group as well as the identity component must be unitary. For the purposes of this work, most groups in question are connected which therefore excludes the possibility of anti-unitary ray representations.

The operators U act on the vectors Ψ_r (for example) in a Hilbert space, and they satisfy the condition (whether unitary or anti-unitary):

$$U^\dagger = U^{-1}.$$

If a group element g_1 transforms the ray $\mathcal{R} \to \mathcal{R}'$, and the element g_2 transforms $\mathcal{R}' \to \mathcal{R}''$ then $g_1 g_2$ transforms $\mathcal{R} \to \mathcal{R}''$. Then the operator U associated with g_1 acts on a vector Ψ in the ray $\mathcal{R}$:

$$U(g_1)\Psi = \Psi',$$

where Ψ' is in the ray $\mathcal{R}'$. Similarly,

$$U(g_2)\Psi' = \Psi'',$$

so that $U(g_1)U(g_2)\Psi = \Psi''$ and where Ψ'' is in the ray $\mathcal{R}''$. Finally,

$$U(g_1 g_2)\Psi = \tilde{\Psi}'',$$

where $\tilde{\Psi}''$ is a vector in the same ray as Ψ'', but it is not necessarily the same vector. Two vectors in the same ray can differ only by a phase, and so:

$$U(g_1)U(g_2) = e^{i\phi(g_1,g_2)}U(g_1 g_2). \tag{2.9}$$

This is a composition rule for a *projective representation*. This is similar to a linear representation, other than the phase in the composition rule. It is a "representation up to a phase", ϕ, which must satisfy certain conditions [78]. Specifically, from associativity:

$$\begin{aligned} & U(g_1)(U(g_2)U(g_3)) = (U(g_1)U(g_2))U(g_3) \\ \Rightarrow \quad & U(g_1)e^{i\phi(g_2,g_3)}U(g_2 g_3) = e^{i\phi(g_1,g_2)}U(g_1 g_2)U(g_3) \\ \Rightarrow \quad & e^{i\phi(g_2,g_3)}e^{i\phi(g_1,g_2 g_3)}U(g_1 g_2 g_3) = e^{i\phi(g_1,g_2)}e^{i\phi(g_1 g_2,g_3)}U(g_1 g_2 g_3), \end{aligned}$$

ϕ must satisfy the following condition:

$$\phi(g_2,g_3) + \phi(g_1, g_2 g_3) = \phi(g_1,g_2) + \phi(g_1 g_2, g_3). \tag{2.10}$$

If the phase ϕ can be written in the form:

$$\phi(g_1,g_2) = \gamma(g_1 g_2) - \gamma(g_1) - \gamma(g_2), \tag{2.11}$$

then the left-hand-side of Equation 2.10 becomes:

$$\begin{aligned}\text{LHS} &= \gamma(g_2g_3) - \gamma(g_2) - \gamma(g_3) + \gamma(g_1g_2g_3) - \gamma(g_1) - \gamma(g_2g_3) \\ &= \alpha(g_1g_2g_3) - \gamma(g_1) - \gamma(g_2) - \gamma(g_3),\end{aligned}$$

and the right-hand-side becomes:

$$\begin{aligned}\text{RHS} &= \gamma(g_1g_2) - \gamma(g_1) - \gamma(g_2) + \gamma(g_1g_2g_3) - \gamma(g_1g_2) - \gamma(g_3) \\ &= \alpha(g_1g_2g_3) - \gamma(g_1) - \gamma(g_2) - \gamma(g_3) \\ &= \text{LHS},\end{aligned}$$

and so the condition of Equation 2.10 is satisfied. Therefore if the phase ϕ of a projective representation can be written in this way, then the projective representation can be seen to be equivalent to a linear representation by replacing $U(g)$ with:

$$\rho(g) = U(g)e^{i\gamma(g)},$$

since the composition rule becomes:

$$\begin{aligned}\rho(g_1)\rho(g_2) &= U(g_1)e^{i\gamma(g_1)}U(g_2)e^{i\gamma(g_2)} \\ &= e^{i[\gamma(g_1)+\gamma(g_2)]}e^{i\phi(g_1,g_2)}U(g_1g_2) \\ &= e^{i[\gamma(g_1)+\gamma(g_2)+\phi(g_1,g_2)]}U(g_1g_2) \\ &= e^{i\gamma(g_1g_2)}U(g_1g_2) \\ &= \rho(g_1g_2),\end{aligned}$$

and so ρ satisfies the composition rule for linear representations; the homomorphism condition of Equation 2.1.

Define a set of functions ϕ_i that satisfy Equation 2.11 and are of the form:

$$\phi_i(g_1, g_2) = \phi(g_1, g_2) + \gamma_i(g_1g_2) - \gamma_i(g_1) - \gamma_i(g_2),$$

where $\phi(g_1, g_2)$ is the phase of Equation 2.9 and $\{\gamma_i\}$ is a set of functions. Then $\{\phi_i(g_1, g_2)\}$ is a set called a *two-cocycle*. A trivial two-cocycle is one in which the function $\phi(g_1, g_2) = 0$, and so the projective representation as defined in Equation 2.9 becomes an ordinary linear representation. If a certain symmetry group allows non-trivial two-cocycles, then these projective representations are said to be *intrinsically projective*. In other words, there exist projective representations that cannot be trivially replaced with linear representations as can the projective representations of trivial two-cocycles.

2.2.1 Central Extensions

This section, along with the one following it, largely follows the arguments of Weinberg in [78]. Suppose there is a projective representation U of a Lie group, with general elements g parametrized by a finite set of parameters α^i, as the linear representations are in Section 2.1.3. The parametrized projective representations

will be written as $U(\alpha)$, with the projective representations of another element written as $U(\beta)$. Then the projective representation composition rule (Equation 2.9) becomes:

$$U(\alpha)U(\beta) = e^{i\phi(\alpha,\beta)}U(f(\alpha,\beta)), \tag{2.12}$$

with $f(\alpha,\beta)$ some function of the parameters that satisfies the following condition:

$$f^i(\alpha,0) = f^i(0,\alpha) = \alpha^i, \qquad f^i(0,0) = 0.$$

A power series expansion of this function centred at zero is given, as in Equation 2.5, by:

$$f^i(\alpha,\beta) = \alpha^i + \beta^i + f^i_{jk}\alpha^j\beta^k.$$

A Taylor series expansion of a projective representation is given by:

$$U(\alpha) = I_n + X_i\alpha^i + \frac{1}{2}X_{ij}\alpha^i\alpha^j + ...$$

where X_i and X_{ij} are respectively first- and second-order partial derivatives of U with respect to the parameters. The left-hand-side of Equation 2.12 is therefore, from Equation 2.6:

$$LHS = I_n + X_i\alpha^i + X_i\beta^i + X_i\alpha^iX_j\beta^j + \frac{1}{2}X_{ij}\alpha^i\alpha^j + \frac{1}{2}X_{ij}\beta^i\beta^j + ...$$

The Taylor series expansion of the phase centred at the identity is given by:

$$\phi(\alpha,\beta) = f_{ij}\alpha^i\beta^j + ...$$

where f_{ij} is some constant. The lower order terms vanish due to the condition that $\phi(\alpha,0) = \phi(0,\alpha) = 0$. Therefore the expansion of the right-hand-side of Equation 2.12 is the same as given in Equation 2.7, except for an additional term given by $f_{ij}\alpha^i\beta^j$. Then, still following the method of Section 2.1.3 and subtracting common terms from the left-hand-side and right-hand-side in the expansion of Equation 2.12 yields:

$$\begin{aligned} & X_iX_j\alpha^i\beta^j = X_k{f^k}_{ij}\alpha^i\beta^j + X_{ij}\alpha^i\beta^j + f_{ij}\alpha^i\beta^j \\ \Rightarrow \qquad & X_{ij} = X_iX_j - X_k{f^k}_{ij} + f_{ij}. \end{aligned}$$

But X_{ij} is symmetric, and so:

$$\begin{aligned} X_iX_j - X_k{f^k}_{ij} + f_{ij} &= X_jX_i - X_k{f^k}_{ji} + f_{ji} \\ [X_i,X_j] &= {C^k}_{ij}X_k + C_{ij}, \end{aligned} \tag{2.13}$$

where $C_{ij} := -f_{ij} + f_{ji} = -C_{ji}$. This is identical to Equation 2.8 except for the addition of $\frac{n(n-1)}{2}$ terms C_{ij} which are called the *central charges* of the Lie algebra, as they commute with all the generators of the Lie algebra.

Equation 2.13 must satisfy the Jacobi identity, so:

$$\begin{aligned} 0 &= [X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] \\ &= C^{\ell}{}_{jk} [X_i, X_\ell] + C^{\ell}{}_{ki} [X_j, X_\ell] + C^{\ell}{}_{ij} [X_k, X_\ell] \\ &= C^{\ell}{}_{jk}(C^{m}{}_{i\ell} + C_{i\ell} I_n) + C^{\ell}{}_{ki}(C^{m}{}_{j\ell} + C_{j\ell} I_n) + C^{\ell}{}_{ij}(C^{m}{}_{k\ell} + C_{k\ell} I_n), \end{aligned}$$

which leads to two conditions on the coefficients:

$$C^{\ell}{}_{jk} C^{m}{}_{i\ell} + C^{\ell}{}_{ki} C^{m}{}_{j\ell} + C^{\ell}{}_{ij} C^{m}{}_{k\ell} = 0 \tag{2.14}$$

and

$$C^{\ell}{}_{jk} C_{i\ell} + C^{\ell}{}_{ki} C_{j\ell} + C^{\ell}{}_{ij} C_{k\ell} = 0. \tag{2.15}$$

Equation 2.14 leads to an obvious solution of Equation 2.15:

$$C_{ij} = C^{m}{}_{ij} \theta_m. \tag{2.16}$$

When the coefficients C_{ij} are replaced in this way, then the Lie algebra generators can be redefined to eliminate the central charges of Equation 2.13, by defining new generators:

$$\tilde{X}_i := X_i + \theta_i. \tag{2.17}$$

The Lie algebra of these new generators is of the form of Equation 2.8:

$$\left[\tilde{X}_i, \tilde{X}_j\right] = C^{k}{}_{ij} \tilde{X}_k. \tag{2.18}$$

If Equation 2.16 is the only solution to Equation 2.15, then there are no *intrinsic* central charges - the central charges that appear can always be discarded by the trivial redefinition of the basis of the algebra given by Equation 2.17.

If however there exist solutions of Equation 2.15 other than that given in Equation 2.16, then there exist intrinsic central charges to the Lie algebra. In this case the Lie algebra can be extended to include new generators M_{ij} that have the central charges C_{ij} as their eigenvalues. This is called an (algebraic) *central extension* of the Lie algebra. This extended Lie algebra consists of the set of generators $\{X_i, M_{ij}\}$, $i, j = 1, 2, ..., n$ along with all linear combinations of these generators. The defining relations of the generators are:

$$[X_i, X_j] = C^{k}{}_{ij} X_k + M_{ij}, \tag{2.19}$$

$$[X_i, M_{ij}] = 0. \tag{2.20}$$

If the Lie group representations U are unitary, then the corresponding central generators M_{ij} if they exist are proportional to the identity due to Schur's lemma (cf. Section 2.1.4):

$$M_{ij} = C_{ij} I_n,$$

and

$$[X_i, X_j] = C^k{}_{ij} X_k + C_{ij} I_n.$$

This extended Lie algebra has a corresponding Lie group. If $\mathcal{G}$ is a Lie group which corresponds to the Lie algebra of Equation 2.18, consisting of elements $\{e^{a^i X_i}\}$ where a^i is an n-tuple, and $\widetilde{\mathcal{G}}$ is a Lie group which corresponds to the Lie algebra of Equations 2.19 and 2.20, consisting of elements $\{e^{a^i X_i}, e^{b^{ij} M_{ij}}\}$, then $\widetilde{\mathcal{G}}$ is called the *central extension* of $\mathcal{G}$. This will be referred to as the "algebraic" central extension of $\mathcal{G}$.

A given Lie group $\mathcal{G}$ may or may not admit an algebraic central extension. If it does not admit an algebraic central extension then there might still be intrinsically projective representations of the group indicating that the group is not its own central extension. Indeed, quite apart from the algebraic central extension, there is the possibility of a topological central extension to $\mathcal{G}$. This possibility is discussed in Section 2.2.2. Central extensions are important since whenever there exist intrinsic projective representations of a group, these projective representations are isomorphic to linear representations of a central extension of the group:

$$U(\mathcal{G}) \cong \rho(\widetilde{\mathcal{G}}).$$

However, these linear representations $\rho(\widetilde{\mathcal{G}})$ are not necessarily matrix representations, even if $\mathcal{G}$ is a matrix group (cf. [37]).

2.2.2 Universal Covering Groups

A given Lie algebra $\mathfrak{g}$ has a set $\{\mathcal{G}_i\}$ of groups associated with it, arrived at through the exponential map. When two groups have an isomorphic Lie algebra, the groups are said to be locally isomorphic to one another. In the set $\{\mathcal{G}_i\}$, which is the complete set of groups that are locally isomorphic to one another, there always exists a group which is simply connected. This Lie group is called the *universal covering group*. The universal covering group is the universal covering group of a group $\mathcal{G}$ in a singular sense, but also of the set of groups $\{\mathcal{G}_i\}$ in a collective sense. It is denoted by $\overline{\mathcal{G}}$ in the context of being the universal covering group of the matrix group $\mathcal{G}$ (as well as any group that may be locally isomorphic to $\mathcal{G}$). It will be referred to as a *topological* central extension of $\mathcal{G}$.

The matrix group $\mathcal{G}$ can be written as a quotient group, in terms of its universal covering group:

$$\mathcal{G} = \overline{\mathcal{G}}/\mathcal{H}.$$

Here, $\mathcal{H}$ is a normal subgroup of $\overline{\mathcal{G}}$ (see Section 2.3 for the definition of a normal subgroup). In fact, it is the first homotopy group of $\overline{\mathcal{G}}$. There exists a smooth homomorphism p, called a *covering homomorphism*, that takes the universal covering group back to the group:

$$p : \overline{\mathcal{G}} \to \mathcal{G}.$$

The kernel of this homomorphism is the first homotopy group $\mathcal{H}$. The first homotopy group is a discrete group. If it is trivial, then the group $\mathcal{G}$ is necessarily simply connected and therefore it is its own universal covering group. Otherwise, the elements of $\mathcal{H}$ commute with everything and so provide additional central elements to the extended group.

A given group $\mathcal{G}$ may admit an algebraic central extension and/or it may have a topological central extension. If a group $\mathcal{G}$ does not admit an algebraic extension, then the universal covering group $\overline{\mathcal{G}}$ is also the full central extension $\widehat{\mathcal{G}}$ of $\mathcal{G}$:

$$\widehat{\mathcal{G}} = \overline{\mathcal{G}}. \tag{2.21}$$

If $\mathcal{G}$ does admit an algebraic central extension which will be a corresponding group denoted by $\widetilde{\mathcal{G}}$, then Equation 2.21 is no longer true and the universal covering group of $\widetilde{\mathcal{G}}$ must be evaluated in order to arrive at the full central extension $\widehat{\mathcal{G}}$ of $\mathcal{G}$. Note that $\widetilde{\mathcal{G}}$ itself does not admit an *algebraic* central extension, and so the full central extension of $\widetilde{\mathcal{G}}$ is precisely its universal covering group. To reiterate, if a group $\mathcal{G}$ has an algebraic central extension $\widetilde{\mathcal{G}}$ but not a topological central extension, then the full central extension of the group $\widehat{\mathcal{G}}$ is given by $\widetilde{\mathcal{G}}$. If a group $\mathcal{G}$ has a topological central extension $\overline{\mathcal{G}}$ but not an algebraic central extension, then the full central extension $\widehat{\mathcal{G}}$ is given by $\overline{\mathcal{G}}$. In the general case, the full central extension is given by the universal covering group of the algebraically centrally extended group.

2.3 Semidirect Product Groups

A succinct definition of a semidirect product group is given in [38]. Let $\mathcal{N}$ and $\mathcal{K}$ be groups with identities $e_{\mathcal{N}}$ and $e_{\mathcal{K}}$ and elements a and k respectively. Suppose that for all $k \in \mathcal{K}$ there is a homomorphism ϕ of elements $k \in \mathcal{K}$ which is an automorphism ϕ_k of $\mathcal{N}$. In mathematical notation,

$$\phi_k : \mathcal{N} \to \mathcal{N}; \qquad \phi_{k_1}(\phi_{k_2}(a)) = \phi_{k_1 k_2}(a); \qquad k_1, k_2 \in \mathcal{K}.$$

The set of these homomorphisms defines a group (the automorphism group of $\mathcal{N}$), with an inverse of $\phi_k^{-1} = \phi_{k^{-1}}$ and an identity $\phi_{e_{\mathcal{K}}}$. There is now therefore defined a semidirect product group $\mathcal{G}$ of $\mathcal{K}$ and $\mathcal{N}$:

$$\mathcal{G} \cong \mathcal{K} \ltimes \mathcal{N},$$

so long as a suitable group product is defined. Writing the elements $g \in \mathcal{G}$ as tuples of elements of the subgroups, (k, a), this product can be defined as:

$$(k_1, a_1)(k_2, a_2) = (k_1 k_2, a_1 \phi_{k_1}(a_2)). \tag{2.22}$$

Note that

$$\begin{aligned} kak^{-1} &:= (k, e_{\mathcal{N}})(e_{\mathcal{K}}, a)(k, e_{\mathcal{N}})^{-1} \\ &= (k, \phi_k(a))(k^{-1}, e_{\mathcal{N}}) \\ &= (e_{\mathcal{K}}, \phi_k(a)) \\ &= \phi_k(a), \end{aligned}$$

since $\phi_{e_\mathcal{K}}(a) = a$. This means that the homomorphism $\phi_k(a)$ can be replaced by the conjugation of a by k:

$$\phi_k : \mathcal{N} \to \mathcal{N}; \qquad \phi_k(a) \mapsto \tilde{a} = kak^{-1}. \tag{2.23}$$

This in turn means that the automorphisms ϕ_k of $\mathcal{N}$ are in fact *inner* automorphisms of $\mathcal{N}$. It also follows immediately that $\mathcal{N}$ is a normal subgroup of $\mathcal{G}$, since $g\mathcal{N}g^{-1} = \mathcal{N}$. The subgroup $\mathcal{K}$, on the other hand, will be called the *homogeneous subgroup* of $\mathcal{G}$. The homogeneous group must be a subgroup of the automorphism group of the normal subgroup.

Returning now to the group product of Equation 2.22; this product is associative, since

$$\begin{aligned}
[(k_1, a_1)(k_2, a_2)]\,(k_3, a_3) &= (k_1k_2, a_1\phi(k_1)(a_2))(k_3, a_3) \\
&= (k_1k_2k_3, a_1\phi(k_1)(a_2)\phi(k_1k_2)(a_3)) \\
&= (k_1k_2k_3, a_1k_1a_2k_1^{-1}k_1k_2a_3k_2^{-1}k_1^{-1}) \\
&= (k_1k_2k_3, a_1k_1a_2k_2a_3k_2^{-1}k_1^{-1})
\end{aligned}$$

and

$$\begin{aligned}
(k_1, a_1)\,[(k_2, a_2)(k_3, a_3)] &= (k_1, a_1)(k_2k_3, a_2\phi(k_2)(a_3)) \\
&= (k_1k_2k_3, a_1\phi(k_1)(a_2\phi(k_2)(a_3))) \\
&= (k_1k_2k_3, a_1k_1a_2k_2a_3k_2^{-1}k_1^{-1}).
\end{aligned}$$

The group has an identity given by $(e_\mathcal{K}, e_\mathcal{N})$, and an inverse which is given by $(k^{-1}, \phi(k^{-1})(a^{-1}))$, since:

$$\begin{aligned}
(k, a)(k^{-1}, \phi(k^{-1})(a^{-1})) &= (kk^{-1}, a\phi(k)(\phi(k^{-1})(a^{-1}))) \\
&= (e_\mathcal{K}, akk^{-1}a^{-1}k^{-1}k) \\
&= (e_\mathcal{K}, e_\mathcal{N}).
\end{aligned}$$

Note that the set of all tuples $(e_\mathcal{K}, a)$ is isomorphic to $\mathcal{N}$ and the set of all tuples $(k, e_\mathcal{N})$ is isomorphic to $\mathcal{K}$. Since the group product is associative and there is an identity and an inverse, $\mathcal{G}$ is indeed a group.

The following are some other properties of the semidirect product group $\mathcal{G}$:

- $\mathcal{G} = \mathcal{K} \cup \mathcal{N}$, since $g = (k, a) = (k, e_\mathcal{N})(e_\mathcal{K}, a)$.
- $e_\mathcal{G} := (e_\mathcal{K}, e_\mathcal{N}) = \mathcal{K} \cap \mathcal{N}$.
- Every element of $\mathcal{G}$ may be written as a product of an element of $\mathcal{K}$ and an element of $\mathcal{N}$. In other words, $\mathcal{G} = \mathcal{K}\mathcal{N}$.
- $\mathcal{K}$ is isomorphic to $\mathcal{G}/\mathcal{N}$.

According to [66] the group $\mathcal{K} \ltimes \mathcal{N}$, constructed this way as a semidirect product group, is a separable locally compact topological group with respect to the product topology, if the homomorphism ϕ has suitable topological properties.

If the homomorphism ϕ maps elements a to itself: $\phi(k)(a) \mapsto \tilde{a} = a$ (or in other words, $kak^{-1} = a$) then the semidirect product becomes a direct product of $\mathcal{K}$ and $\mathcal{N}$:

$$\mathcal{G} = \mathcal{K} \times \mathcal{N} \tag{2.24}$$

This is also the case if the homogeneous subgroup $\mathcal{K}$ is itself normal, as well as $\mathcal{N}$. Indeed, the subgroup $\mathcal{K}$ in the direct product in Equation 2.24 is necessarily normal.

2.3.1 Little Groups and Stabiliser Groups

Consider the *group action* of a group, $\mathcal{G}$ acting on a set X. This is defined by elements $g \in \mathcal{G}$ acting on elements $x \in X$, such that the image of g acting on x is in X: $g \cdot x \in X$. Multiple actions of elements of $\mathcal{G}$ acting on x are associative: $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$, and the identity of $\mathcal{G}$ acting on x yields x: $e_g \cdot x = x$.

The *orbit* of x is defined as the set of elements in X that can be possibly obtained through elements of $\mathcal{G}$ acting on x, and is denoted by $\mathcal{G} \cdot x$. In mathematical notation, this is:

$$\mathcal{G} \cdot x = \{g \cdot x \in X : g \in \mathcal{G}\}.$$

If there are elements other than the identity of $\mathcal{G}$, that yield the input x as an image of the group action, then the set of these elements of $\mathcal{G}$ forms a subgroup of $\mathcal{G}$, called an *isotropy group*. This is denoted by $\mathcal{G}^x$. In mathematical notation,

$$\mathcal{G}^x = \{g \in \mathcal{G} : g \cdot x = x, x \in X\}.$$

In the case of a semidirect product group, consider the group action of the homogeneous group, $\mathcal{K}$, acting on the unitary dual[3] $\widehat{\mathcal{N}}$ of the normal subgroup, $\mathcal{N}$. This unitary dual space consists of the set of equivalence classes $[\xi]$ of continuous unitary irreducible representations ξ of elements $a \in \mathcal{N}$ (cf. ch. 16 §1 of [4]). These representations act on states $|\psi\rangle$ in a Hilbert space $\mathtt{H}^\xi$ in the following way:

$$\xi(a) : \mathtt{H}^\xi \to \mathtt{H}^\xi : |\psi\rangle \mapsto |\tilde{\psi}\rangle = \xi(a)\,|\psi\rangle\,.$$

The homogeneous group has representations $\sigma(k)$ which act in a similar manner on the Hilbert space $\mathtt{H}^\sigma$. Elements $k \in \mathcal{K}$ act on the representations ξ:

$$k : \widehat{\mathcal{N}} \to \widehat{\mathcal{N}} : \xi \mapsto \tilde{\xi} = k \cdot \xi.$$

This action of elements $k \in \mathcal{K}$ acting on representations ξ (and so by extension, on the equivalence classes of representations, $[\xi]$) is defined to be:

$$k \cdot \xi = \xi(kak^{-1}). \tag{2.25}$$

[3]Though it uses the same notation, this should not be confused with central extensions of the group.

Two representations ξ and $\tilde{\xi}$ are equivalent if:

$$\tilde{\xi} = U\xi U^{-1}, \tag{2.26}$$

U being some unitary operator. Therefore, the equivalence class of ξ is defined to be:

$$[\xi] = \{\tilde{\xi} = U\xi U^{-1}\}; \tag{2.27}$$

it is the set of all representations ξ that can be reached under conjugation of operators U. $\widehat{\mathcal{N}}$ is partitioned by its equivalence classes $[\xi]$, but the action of k can take one equivalence class into another. Therefore each equivalence class has orbit:

$$\begin{aligned}\mathcal{K} \cdot [\xi] &= \{k \cdot [\xi] \in \widehat{\mathcal{N}}, k \in \mathcal{K}\} \\ &= \{[k \cdot \xi] \in \widehat{\mathcal{N}}, k \in \mathcal{K}\},\end{aligned}$$

since for $\xi, \tilde{\xi} \in [\xi]$:

$$\begin{aligned}k \cdot \tilde{\xi}(a) &= \tilde{\xi}(k^{-1}ak) \\ &= U\xi(k^{-1}ak)U^{-1} \\ &= U(k \cdot \xi(a))U^{-1}.\end{aligned}$$

The orbit is written as $[\mathcal{K} \cdot \xi]$. On each orbit, there is defined a *little group*, $\mathcal{K}^\xi$ such that:

$$k \cdot [\xi] = [\xi], \tag{2.28}$$

for any $k \in \mathcal{K}^\xi$. This little group is a specific case of an isotropy group. Equation 2.28 in turn defines a fixed-point condition for ξ: since $[\xi]$ is defined by Equation 2.27, it follows that the fixed-point condition for the little group $\mathcal{K}^\xi$ is:

$$k \cdot \xi = \tilde{\xi} = \rho_k \xi {\rho_k}^{-1}, \tag{2.29}$$

since $\widehat{\mathcal{N}}$ consists of the set of equivalence classes of ξ, and where ρ is a unitary operator that depends on k in some way in the group action. The fixed-point condition means the little group consists of the elements of $\mathcal{K}$ that act on representations ξ by moving them to another representation within its equivalence class.

For $k_1, k_2 \in \mathcal{K}^\xi$, two successive group actions on ξ yields:

$$(k_1 k_2) \cdot \xi = \rho_{k_1 k_2} \xi \rho^{-1}_{k_1 k_2},$$

from Equation 2.29. But if the actions are taken one at a time, the two successive group actions yield:

$$(k_1) \cdot (k_2 \cdot \xi) = \rho_{k_1} \rho_{k_2} \xi \rho^{-1}_{k_2} \rho^{-1}_{k_1}.$$

Equating the right hand sides of these two equations gives us the equation:

$$\rho_{k_1k_2}\xi\rho_{k_1k_2}^{-1} = \rho_{k_1}\rho_{k_2}\xi\rho_{k_2}^{-1}\rho_{k_1}^{-1}.$$

Defining a new operator Ω in the following manner:

$$\Omega_{k_1k_2} := \rho_{k_2}^{-1}\rho_{k_1}^{-1}\rho_{k_1k_2}$$

allows the following condition to be set:

$$\Omega_{k_1k_2}\xi = \xi\Omega_{k_1k_2}.$$

ξ is irreducible, so Schur's lemma demands Ω to be a scalar multiple of the identity operator:

$$\Omega_{k_1k_2} = \phi(k_1, k_2) \times I.$$

Therefore,

$$\rho_{k_1k_2} = \phi(k_1, k_2)\rho_{k_1}\rho_{k_2}. \tag{2.30}$$

This is a homomorphism condition for a *projective representation*, and so $\rho(k)$ is a projective representation of elements k in the little group $\mathcal{K}^\xi$ (therefore the subscript notation will be dropped in favour of a bracket notation appropriate for representations).

The semidirect product of a little group with the normal subgroup defines a new semidirect product group, known as the stabiliser group, $\mathcal{G}^\xi$:

$$\mathcal{G}^\xi = \mathcal{K}^\xi \ltimes \mathcal{N}. \tag{2.31}$$

Sometimes in the literature the little group $\mathcal{K}^\xi$ is referred to as a "stabiliser group" or an "isotropy group". It is referred to here as a little group so as to avoid confusion with the stabiliser group defined in Equation 2.31 and to indicate that it is a specific kind of isotropy group.

Now, ρ is a projective extension of the representation ξ to $\mathcal{G}^\xi$, such that:

$$\rho(g) : \mathtt{H}^\xi \to \mathtt{H}^\xi, \qquad \forall g \in \mathcal{G}^\xi;$$

the projective extension representations act on the Hilbert space of the normal subgroup. Clearly, the representations ρ of elements of the normal subgroup are equivalent to the representations ξ of those elements:

$$\rho|_{\mathcal{N}} \cong \xi.$$

In other words, $\rho(a) \cong \xi(a)$ for $a \in \mathcal{N}$.

Equation 2.25 can be modified to incorporate ρ:

$$k \cdot \xi(a) = \rho(kak^{-1}),$$

and so the fixed-point condition becomes:

$$k \cdot \xi(a) = \rho(k)\xi(a)\rho^{-1}(k).$$

This condition must be true at every point of the unitary dual space $\widehat{\mathcal{N}}$. Therefore, the little groups of semidirect product groups are comprised of elements:

$$\mathcal{K}^\xi = \{k \in \mathcal{K} : k \cdot \xi = \rho(k)\xi\rho^{-1}(k)\}. \tag{2.32}$$

The representations of elements of these little groups are just the ordinary representations of the homogeneous subgroup:

$$\sigma(k) : \mathtt{H}^{\sigma^\xi} \to \mathtt{H}^{\sigma^\xi}, k \in \mathcal{K}^\xi,$$

where the Hilbert space $\mathtt{H}^{\sigma^\xi}$ is the Hilbert space of the homogeneous group $\mathtt{H}^\sigma$ restricted to the little group. The representations τ^ξ of the stabiliser groups are (from [55]):

$$\tau^\xi = \sigma \otimes \rho : \mathtt{H}^{\tau^\xi} \to \mathtt{H}^{\tau^\xi}, \tag{2.33}$$

where $\mathtt{H}^{\tau^\xi} = \mathtt{H}^{\sigma^\xi} \otimes \mathtt{H}^\xi$.

2.3.2 Abelian Normal Subgroups

Representations of abelian groups are irreducible if and only if they are one-dimensional. Therefore, in an irreducible representation, the representation of an element $a \in \mathcal{N} \cong \mathbb{R}^n$ can be identified with the trace of its matrix realisation - the character, $\chi_c(a)$ - which acts on the Hilbert space: $\mathtt{H}^\xi = \mathbb{C}$:

$$\chi : a \in \mathbb{R}^n \to \mathtt{H}^\xi = \mathbb{C} \tag{2.34}$$

$$\chi(a) : \mathbb{C} \to \mathbb{C}, \qquad \chi(a) = e^{ic_i a^i}, \tag{2.35}$$

where c_i is an n-tuple labelling the characters and a^i is an n-tuple which is the group element, $a^i \equiv a$.

In the case of semidirect product groups where the normal subgroup is abelian, the fixed-point condition defining the little groups (Equation 2.32), simplifies significantly. For abelian groups the equivalence classes consist of one representation only (cf. Equation 2.26). Therefore the unitary dual space $\widehat{\mathcal{N}}$ is now made up of the set of characters χ_c of elements $a \in \mathcal{N}$. The fixed-point condition that defines little groups under the action of $\mathcal{K}$ is therefore:

$$k \cdot \xi = \xi. \tag{2.36}$$

Therefore the little groups are defined to be:

$$\mathcal{K}^\xi = \{k \in \mathcal{K} : k \cdot \chi_c = \chi_c\}. \tag{2.37}$$

The stabiliser groups are still given by Equation 2.31 and the representations τ^ξ of elements of the stabiliser groups for abelian normal subgroups are given by (cf. Equation 2.33):

$$\tau^\xi = \sigma \otimes \xi, \tag{2.38}$$

and the Hilbert space is given by:

$$\begin{aligned} \mathtt{H}^{\tau^\xi} &= \mathtt{H}^{\sigma^\xi} \otimes \mathtt{H}^\xi \\ &= \mathtt{H}^{\sigma^\xi} \otimes \mathbb{C}. \end{aligned} \tag{2.39}$$

2.4 Mackey Induction Method

For further reading on the Mackey induction method, see [63, 64, 65, 66] and [69].

In a semidirect product group construction, if the stabiliser group is equivalent to the full semidirect product group - $\mathcal{G} \cong \mathcal{G}^\xi$ - then the representations of the semidirect product group are simply the representations of the stabiliser group: $\tau = \tau^\xi$. These representations act on the Hilbert space $\mathtt{H}^\tau \cong \mathtt{H}^{\tau^\xi} = \mathtt{H}^{\sigma^\xi} \otimes \mathtt{H}^\xi$. Otherwise, each class of the stabiliser subgroup of the semidirect product group $\mathcal{G}$ must be studied on a case-by-case basis. Mackey's Induction method must then be used to determine the representations and the Hilbert space on which they operate, as follows.

Following [69], let μ be an invariant measure on $\mathbb{X} := \mathcal{G}/\mathcal{G}^\xi$; the quotient space being comprised of cosets x. Then the Hilbert space on which the induced representations operate is:

$$\begin{aligned}\mathtt{H}^\tau &\cong \mathtt{H}^{\sigma^\xi} \otimes \mathcal{L}^2(\mathbb{X}, \mathtt{H}^\xi, \mu),\\ &= \mathcal{L}^2(\mathcal{G}/\mathcal{G}^\xi, \mathtt{H}^{\tau^\xi}, \mu).\end{aligned}$$

Define a coset representative function $\Theta(x) \in \mathcal{G}$, such that Θ is a Borel function from $\mathbb{X} \to \mathcal{G}$. Then,

$$x = \Theta(x)\mathcal{G}^\xi. \tag{2.40}$$

Multiplying this equation on the left by g^{-1} yields:

$$g^{-1}x = g^{-1}\Theta(x)\mathcal{G}^\xi.$$

But $g^{-1}x = \Theta(g^{-1}x)\mathcal{G}^\xi$, from Equation 2.40. Therefore,

$$\Theta(g^{-1}x) = g^{-1}\Theta(x)h$$

for some $h \in \mathcal{G}^\xi$. By multiplying both sides of this equation by $\Theta^{-1}(x)g$, this implies that:

$$\Theta^{-1}(x)g\Theta(g^{-1}x) \in \mathcal{G}^\xi. \tag{2.41}$$

In the context of the Poincaré group, this is a "Wigner Rotation".

From [69], the induced representations τ of the semidirect product group $\mathcal{G}$ are:

$$(\tau(g)\psi)(x) = \tau^\xi(\Theta(x)^{-1}g\Theta(g^{-1}x))\psi(g^{-1}x), \tag{2.42}$$

where $g \in \mathcal{G}$, $x \in \mathcal{G}/\mathcal{G}^\xi$, $\psi \in \mathtt{H}^\tau = L^2(\mathcal{G}/\mathcal{G}^\xi, \mathtt{H}^{\tau^\xi}, \mu)$. Let $\Phi(x) := (\tau(g_2)\psi)(x), g_2 \in \mathcal{G}$. Then for $g_1 \in \mathcal{G}$,

$$\begin{aligned}&(\tau(g_1)(\tau(g_2)\psi))(x) =\\ &\quad= (\tau(g_1)\Phi)(x)\\ &\quad= \tau^\xi(\Theta^{-1}(x)g_1\Theta({g_1}^{-1}x))\Phi({g_1}^{-1}x)\\ &\quad= \tau^\xi(\Theta^{-1}(x)g_1\Theta({g_1}^{-1}x))\tau^\xi(\Theta^{-1}({g_1}^{-1}x)g_2\Theta({g_2}^{-1}{g_1}^{-1}x))\psi({g_2}^{-1}{g_1}^{-1}x)\\ &\quad= \tau^\xi(\Theta^{-1}(x)g_1g_2\Theta((g_1g_2)^{-1}x))\psi((g_1g_2)^{-1}x)\\ &\quad= (\tau(g_1g_2)\psi)(x),\end{aligned}$$

and so the group product is preserved:

$$(\tau(g_1)(\tau(g_2)\psi))(x) = (\tau(g_1 g_2)\psi)(x).$$

If $(g_1)^{-1} = g_2$ then from this same proof it follows that the inverse of $(\tau(g_1)\psi)(x)$ is $(\tau((g_1)^{-1})\psi)(x)$:

$$(\tau^{-1}(g)\psi)(x) = (\tau(g^{-1})\psi)(x).$$

Then

$$\begin{aligned}(\tau^{-1}(g)\psi)(x) &= (\tau^{\xi})^{-1}(\Theta(x)^{-1}g\Theta(g^{-1}x))\psi(g^{-1}x) \\ &= (\tau^{\xi})^{\dagger}(\Theta(x)^{-1}g\Theta(g^{-1}x))\psi(g^{-1}x),\end{aligned}$$

where the second line follows if and only if τ^{ξ} itself is unitary. Since, from Equation 2.33, $\tau^{\xi} = \sigma \otimes \rho$, where ρ is the projective extension of ξ to elements $k \in \mathcal{K}^{\xi}$, this means that τ will be a unitary representation of $\mathcal{G}$ if and only if σ and ξ are unitary representations of the groups $\mathcal{K}$ and $\mathcal{N}$ respectively.

2.5 The Translation Group

The translation groups can be built up as direct products of smaller translation groups:

$$\mathcal{T}(n) = \mathcal{T}(n-1) \times \mathcal{T}(1).$$

A direct product is a particular kind of semidirect product, and so the Mackey induction theory can be used to identify the unitary irreducible representations of $\mathcal{T}(n)$, from the unitary irreducible representations of the subgroups.

Write the elements $g \in \mathcal{T}(n)$ as ordered pairs of the abstract elements $k \in \mathcal{T}(n-1)$ and $a \in \mathcal{T}(1)$:

$$\begin{aligned}g &= (k, a) \\ &= (b^i, a),\end{aligned}$$

where $i = 1, 2, ..., n-1$ and so $k \equiv b^i$ is an $(n-1)$-tuple. Then elements of $\mathcal{T}(n-1)$ can be written as $(b^i, 0)$ and elements of $\mathcal{T}(1)$ can be written as $(0, a)$. The group product for general elements $g, g' \in \mathcal{T}(n)$ is given by:

$$(b^i, a)((b^i)', a') = (b^i + (b^i)', a + a').$$

The inverse of $g = (b^i, a)$ is $g^{-1} = (-b^i, -a)$. Then,

$$\begin{aligned}kak^{-1} &= (b^i, 0)(0, a)(-b^i, 0) \\ &= (b^i, a)(-b^i, 0) \\ &= (0, a) \\ &= a.\end{aligned} \tag{2.43}$$

A similar calculation shows that $aka^{-1} = k$, and so both subgroups are normal. This is to be expected, as it is the condition that makes a semidirect product a direct product.

Although both subgroups are normal, take $\mathcal{T}(n-1)$ to be the homogeneous group with $\mathcal{T}(1)$ taking the role of $\mathcal{N}$ in Section 2.3. The irreducible representations of an element a of the one dimensional translation group are given by the characters, χ_c:

$$\xi(a) := \chi_c(a) = e^{ica},$$

where c labels these characters. Note that the representation is one dimensional. This is due to the fact that irreducible representations of abelian groups must be one dimensional due to Schur's lemma (cf. Section 2.1.4) and this has nothing to do with the fact that the translation group itself is also one dimensional. The little groups are defined by Equation 2.37, as the elements $k \in \mathcal{T}(n-1)$ such that:

$$k \cdot \xi = \xi.$$

In the group action, the elements of the homogeneous group $k = b^i$ act on the representations of $\mathcal{T}(1)$ in the following manner, from Equation 2.25:

$$\begin{aligned} k \cdot \xi &= \xi(kak^{-1}) \\ &= \xi(a), \end{aligned}$$

since $kak^{-1} = a$ from Equation 2.43. This means that the representations of $\mathcal{T}(1)$ remain invariant under action of any of the elements of the homogeneous group. Therefore there is only one little group: the full homogeneous group.

As the stabiliser groups are defined to be the semidirect product of each little group with the normal subgroup, in this case there is only one stabiliser group $\mathcal{G}^\xi = \mathcal{T}(n-1) \ltimes \mathcal{T}(1)$ which is equivalent to the full semidirect product group $\mathcal{G} = \mathcal{T}(n)$. When this is the case, the Mackey induction method does not need to be used. The representations of the homogeneous group - like the representations of any translation group - are also given by the characters:

$$\sigma(k) := \chi_{d_i}(b^i) = e^{id_i b^i},$$

where b_i is an $(n-1)$-tuple labelling the characters. From Equation 2.38, the representations τ of the semidirect product group are:

$$\begin{aligned} \tau(g) &= \sigma(k) \otimes \xi(n) \\ &= e^{id_i b^i} \otimes e^{ica} \\ &= e^{if_j y^j}, \end{aligned}$$

where y^j is an n-tuple containing k and a: $y^j = (b^1, b^2, ..., b^{n-1}, a)$, and f_j is an n-tuple containing d_i and c: $f_j = (d_1, d_2, ..., d_{n-1}, c)$. The Hilbert space is given by Equation 2.39 as $\mathtt{H}^{\sigma^\xi} \otimes \mathbb{C}$. If it is assumed that the Hilbert space $\mathtt{H}^{\sigma^\xi}$ is $\mathbb{C}$, then $\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$ and the Hilbert space of the group $\mathcal{T}(n)$ is also $\mathbb{C}$. Therefore, due to the principle of mathematical induction, the assumption was correct and $\mathtt{H}^{\sigma^\xi} = \mathbb{C}$.

Changing the notation such that the full semidirect product group has elements a^i, with c_i labelling the representations and now $i = 1, 2, ..., n$, allows the unitary irreducible representations τ of elements a^i of the translation group $\mathcal{T}(n)$ to be written as:

$$\tau(a^i) = e^{ic_i a^i}. \tag{2.44}$$

If for example, $n = 3$, then the unitary irreducible representations of this group, $\mathcal{T}(3)$ are given by Equation 2.44 with $i = 1, 2, 3$ or $i = x, y, z$. In this case the c_i labels are the momentum operators P_1, P_2 and P_3. Furthermore, the direct product of this group $\mathcal{T}(3)$ group with another $\mathcal{T}(3)$ group - the latter one with labels x_1, x_2 and x_3 - results in another translation group, on $\mathbb{R}^2$:

$$\mathcal{T}(6) = \mathcal{T}(3) \otimes \mathcal{T}(3).$$

This group has a trivial Lie algebra since it is abelian. For generators X_i in the Lie algebra of the first translation group and P_i in the second translation group, the Lie algebra is given by:

$$[X_i, X_j] = [P_i, P_j] = [X_i, P_j] = 0. \tag{2.45}$$

However, this Lie algebra admits an algebraic central extension. A central extension of $\mathcal{T}(n)$ is given by the Heisenberg group, $\mathcal{H}(n)$. See Chapter 2.6 for a discussion of this group and Equation 2.48 for the algebraic central extension of the Lie algebra given in Equation 2.45. Also see Section 4.1.1, which discusses how the Heisenberg group contracts to the translation group in the limit $\hbar \to 0$.

2.6 The Weyl-Heisenberg Group

The Weyl-Heisenberg group[4] $\mathcal{H}$ of dimension n is the semidirect product of two translation groups:

$$\begin{aligned} \mathcal{H}(n) &= \mathcal{T}_1(n) \ltimes \mathcal{T}(n+1) \\ &= \mathcal{T}_1(n) \ltimes (\mathcal{T}_2(n) \times \mathcal{T}(1)). \end{aligned}$$

The subscripts 1 and 2 are used simply to differentiate between each translation group of order n. The Hilbert spaces of the groups $\mathcal{T}_1$ and $\mathcal{T}_2$ are $\mathcal{L}^2(\mathbb{R}^n)$ and $\mathcal{L}^2(\mathbb{R}^{n+1})$ respectively.

The Heisenberg group elements $g \in \mathcal{H}(n)$ can be written as ordered pairs of elements of the subgroups, $g = (k, a)$, where $k \in \mathcal{T}_1(n)$ and $a \in \mathcal{T}(n+1)$. Alternatively, the elements of $\mathcal{T}(n+1)$ can themselves be written as ordered pairs, $a = (a^i, c)$ with $a^i \in \mathcal{T}_2(n)$ and $c \in \mathcal{T}(1)$, and $i = 1, 2, ..., n$. Therefore, the Heisenberg group elements can also be written as ordered triples:

$$g = (\alpha^i, a^i, c).$$

[4]For brevity this will simply be called the "Heisenberg group" hereafter.

Here, $\alpha^i \equiv k \in \mathcal{T}_1(n)$. For the purposes of this section, only two elements are needed from each subgroup. Therefore, α^i and β^i will be used as two different elements of $\mathcal{T}_1(n)$, a^i and b^i will be used as two different elements of $\mathcal{T}_2(n)$, and c or c_1 and c_2 will be used as the elements of $\mathcal{T}(1)$.

The Heisenberg group can be realised by 4×4 block matrices:

$$g(\alpha^i, a^i, c) = \begin{pmatrix} 1_n & 0 & 0 & \alpha^i \\ 0 & 1_n & 0 & a^i \\ a_i & -\alpha_i & 1 & 2c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.46}$$

where α^i and a^i represent column vectors, α_i and a_i represent the row vector transpose of the column vectors, and 1_n is an $n \times n$ identity matrix. A simple calculation shows that the Heisenberg group product is, for $(\alpha^i, a^i, c_1), (\beta^i, b^i, c_2) \in \mathcal{H}(n)$:

$$(\alpha^i, a^i, c_1)(\beta^i, b^i, c_2) = (\alpha^i + \beta^i, a^i + b^i, c_1 + c_2 + \tfrac{1}{2}(a \cdot \beta - \alpha \cdot b)),$$

where the dot product indicates the inner product of a row vector and a column vector. The inverse of an element (α^i, a^i, c) is $(-\alpha^i, -a^i, -c)$. Then, for elements $a \equiv (0, a^i, c_1) \in \mathcal{T}(n+1)$:

$$\begin{aligned} gag^{-1} &= (\alpha^i, b^i, c_2)(0, a^i, c_1)(-\alpha^i, -b^i, -c_2) \\ &= (\alpha^i, b^i + a^i, c_1 + c_2 - \tfrac{1}{2}(\alpha \cdot a))(-\alpha^i, -b^i, -c_2) \\ &= (0, a^i, c_1 - \alpha \cdot a) \\ &\in \mathcal{T}(n+1). \end{aligned}$$

This shows that the translation group of higher dimension, $\mathcal{T}(n+1) = \mathcal{T}(n) \otimes \mathcal{T}(1)$, is a normal subgroup of the Weyl-Heisenberg group. A similar calculation shows that $gkg^{-1} \notin \mathcal{T}(n)$ in general and so $\mathcal{T}(n)$ is not a normal subgroup, and the Heisenberg group is not a direct product group. Therefore, take $\mathcal{T}_1(n)$ to be the homogeneous subgroup. As the normal subgroup is a translation group, it has unitary irreducible representations ξ given by Equation 2.44 for elements $a \equiv (0, a^i, c) \in \mathcal{N} = \mathcal{T}(n+1)$, with $i = 1, 2, ..., n$ (however, note the differing notation to Equation 2.44 where the Roman c with a subscript refers to the characters; in this case they refer to elements of the group $\mathcal{T}(1)$).

Elements k of the homogeneous subgroup $\mathcal{T}(n)$ are ordered triples of the form $(\alpha^i, 0, 0)$, with $i = 1, 2, ..., n$. These act on the unitary irreducible representations of the normal subgroup in the following manner, from Equation 2.25:

$$\begin{aligned} k \cdot \xi &= \xi(kak^{-1}) \\ &= \xi[(\alpha^i, 0, 0)(0, a^i, c)(-\alpha^i, 0, 0)] \\ &= \xi[(\alpha^i, a^i, c - \tfrac{1}{2}(\alpha \cdot a))(-\alpha^i, 0, 0)] \\ &= \xi(0, a^i, c - \alpha \cdot a) \\ &= e^{iA_i a^i + iC(c - \alpha \cdot a)}, \end{aligned}$$

where A_i and C are character labels.

The little groups of the Weyl-Heisenberg group are defined by the fixed point condition for abelian normal subgroups, given by Equation 2.36:

$$e^{iA_i a^i + iCc} = e^{iA_i a^i + iC(c-\alpha\cdot a)}$$
$$\Rightarrow Cc = C(c - \alpha \cdot a).$$

There are two cases for this fixed point condition to be satisfied:

1. $C \equiv 0$ and so the little group is the entire homogeneous subgroup. This case corresponds to classical mechanics. The unitary irreducible representations are calculated in Section 2.6.2.1

2. $C \neq 0, \alpha \cdot a = 0$ and so the little group is restricted to just the identity element of the homogeneous subgroup. This case corresponds to quantum mechanics. The unitary irreducible representations are calculated in Section 2.6.2.2

2.6.1 Heisenberg Algebra

Representations of the algebra of the Heisenberg group can be calculated directly from the (non-unitary) matrix realisation in Equation 2.46. The generators of the algebra are given by the derivative of $g(\alpha^i, a^i, c)$ with respect to each parameter at the identity. There are three types of parameter, and hence there are three types of generator, making a total of $2n + 1$ generators (one for each parameter). The generators are given by:

$$X^i = i\frac{\partial g}{\partial a^i}\bigg|_{a^i=0}, \qquad P^i = i\frac{\partial g}{\partial \alpha^i}\bigg|_{\alpha^i=0} \qquad I = i\frac{\partial g}{\partial c}\bigg|_{c=0}.$$

In $n = 1$ dimensions, the matrix realisations of these generators are:

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad P = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.47a}$$

$$I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2i \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.47b}$$

The generalisation to n dimensions is easily done. For example, in the general P^j generator there will be an i in the j^{th} row of the last column, and a $-i$ in the $(n + j)^{\text{th}}$ column of the second last row; all other entries being zero. These generators satisfy the following commutation relations:

$$[X^i, P_j] = i\delta^i{}_j I, \tag{2.48}$$

with all other combinations of generators commuting amongst each other. A general element $\mathfrak{Z}$ of the Heisenberg Lie algebra can then be written as:

$$\mathfrak{Z} = i(a_i X^i + \alpha_i P^i + cI). \tag{2.49}$$

This is further refined in Equation 2.62.

2.6.2 Representations of the Heisenberg Group

2.6.2.1 Classical Case

In the classical case, $I \equiv 0$ and the little group consists of the entire homogeneous subgroup. As the stabiliser group is the semidirect product of the little group with the normal subgroup, in this case the stabiliser group is the entire Weyl-Heisenberg group. Therefore, the Mackey induction method does not need to be used, and the unitary irreducible representations τ of the group are, from Equation 2.38, the tensor product of the unitary irreducible representations $\sigma = e^{iX_i\alpha^i}$ of the homogeneous group with the unitary irreducible representations $\xi = e^{iP_ia^i}$ of the normal subgroup:

$$\begin{aligned}\tau &= \sigma \otimes \xi \\ &= e^{iX_i\alpha^i + iP_ia^i}.\end{aligned} \tag{2.50}$$

These representations act on the Hilbert space:

$$\begin{aligned}\mathtt{H}^{\tau} &= \mathtt{H}^{\sigma^{\xi}} \otimes \mathtt{H}^{\xi} \\ &= \mathbb{C} \otimes \mathbb{C} \\ &= \mathbb{C}.\end{aligned}$$

Note that in the tensor product, the dimension of the product is equal to the product of the dimensions.

Also note that the representations of Equation 2.50 are the representations of a translation group of order $2n$. That is $\mathcal{T}(2n)$, where configuration space consists of an N-tuple f^N, $N = 1, 2, ..., 2n$, $f^i = \alpha^i$ and $f^{i+n} = a^i$. In other words, the unitary irreducible representation τ is a mapping from $\mathbb{R}^{2n}$ to the circle group [30]. This is an abelian group, and so the Lie algebra is trivial.

2.6.2.2 Quantum Case

In the quantum case, $I \neq 0$ and the little group consists of solely the identity element of the homogeneous group. The stabiliser group therefore is the semidirect product of this element with the normal subgroup:

$$\mathcal{G}^{\xi} = \{0\} \ltimes \mathcal{T}(n+1) \cong \mathcal{T}(n+1).$$

The unitary irreducible representations are defined by the construction of Equation 2.42. Define $\mathbb{X}$ to be the quotient space of the Weyl-Heisenberg group and the stabiliser group: $\mathbb{X} = \mathcal{H}(n)/\mathcal{T}(n+1) \cong \mathcal{T}(n)$. Then Θ is the coset representative function that takes elements of $x \in \mathbb{X}$ to $\mathcal{H}(n)$:

$$\Theta : x \in \mathcal{T}(n) \to \mathcal{H}(n).$$

Choose $\Theta(x) = (\alpha^i, 0, 0)$. Then $x = (\alpha^i, 0, 0)\mathcal{N}$, where $\mathcal{N} = \mathcal{T}(n+1)$. For $g^{-1} = (-\beta^i, -a^i, -c)$,

$$\begin{aligned} g^{-1}x &= (-\beta^i, -a^i, -c)(\alpha^i, 0, 0)\mathcal{N} \\ &= (\alpha^i - \beta^i, -a^i, -c - \tfrac{1}{2}a \cdot \alpha)\mathcal{N} \\ &= (\alpha^i - \beta^i, 0, 0)(0, -a^i, -c - a \cdot \alpha - \tfrac{1}{2}a \cdot \beta)\mathcal{N} \\ &= (\alpha^i - \beta^i, 0, 0)\mathcal{N} \\ \Rightarrow \Theta(g^{-1}x) &= (\alpha^i - \beta^i, 0, 0), \end{aligned}$$

and

$$\begin{aligned} \Theta^{-1}(x) &= \Theta(x^{-1}) \\ &= (-\alpha^i, 0, 0). \end{aligned}$$

Therefore,

$$\begin{aligned} \Theta^{-1}(x)g\Theta(g^{-1}x) &= (-\alpha^i, 0, 0)(\beta^i, a^i, c)(\alpha^i - \beta^i, 0, 0) \\ &= (\beta^i - \alpha^i, a^i, c + \tfrac{1}{2}\alpha \cdot a)(\alpha^i - \beta^i, 0, 0) \\ &= (0, a^i, c + \alpha \cdot a - \tfrac{1}{2}a \cdot \beta) \end{aligned}$$

From Equation 2.42, the representations of the quantum case of the Heisenberg group are given by:

$$\begin{aligned} (\tau(g)\psi)(x) &= \tau^\xi(\Theta(x)^{-1}g\Theta(g^{-1}x))\psi(g^{-1}x) \\ &= \tau^\xi(0, a^i, c + \alpha \cdot a - \tfrac{1}{2}a \cdot \beta)\psi((\alpha^i - \beta^i, 0, 0)\mathcal{N}) \end{aligned}$$

The representations τ^ξ are the representations of the stabiliser group, $\mathcal{T}(n+1)$. From Equation 2.38, these representations are the tensor product of the representations σ of the homogeneous group and the representations of the little group, ξ. In this case, this reduces to simply the representations of the little group. As elements x of the quotient space $\mathbb{X}$ are dependent only upon the parameter $\alpha^i \in \mathcal{T}_1(n)$, $x \equiv (\alpha^i, 0, 0)$ will be replaced by simply α^i. This means that $g^{-1}x$ will be replaced by $\alpha^i - \beta^i$. Then the representations are:

$$\begin{aligned} (\tau(g)\psi)(\alpha^i) :&= \tilde{\psi}(\alpha^i) \\ &= \tau^\xi(0, a^i, c + \alpha \cdot a - \tfrac{1}{2}a \cdot \beta)\psi(\alpha^i - \beta^i) \\ &= e^{iA_i a^i + iC(c + \alpha \cdot a - \frac{1}{2}a \cdot \beta)}\psi(\alpha^i - \beta^i), \end{aligned}$$

where A_i and C are the character labels. Now,

$$\psi(\alpha^i - \beta^i) = e^{-\beta^i \frac{\partial}{\partial \alpha^i}}\psi(\alpha^i),$$

by a simple Taylor expansion of $\psi(\alpha^i - \beta^i)$ centred at α^i (this can be seen by evaluating the left-hand-side and right-hand-side of the equation to a few terms - each side of the equation matches the other term by term) and so

$$\tilde{\psi}(\alpha^i) = e^{iA_i a^i + iC(c + \alpha \cdot a - \frac{1}{2}a \cdot \beta)} e^{i\beta^i i \frac{\partial}{\partial \alpha^i}}\psi(\alpha^i). \tag{2.51}$$

A simple calculation[5] shows that the Baker-Campbell-Hausdorff formula for the Heisenberg algebra, with non-commuting generators X and Y, is given exactly by:

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{2.52}$$

In fact, this is true for any set of operators $\{X, Y\}$ which all commute with their commutator: $[X,[X,Y]] = [Y,[X,Y]] = 0$, which means Equation 2.52 can be applied to Equation 2.51, which is made up of linear combinations of operators which all commute with their commutators. Setting $X = i[A_i a^i + C(c + \alpha\cdot a - \frac{1}{2}a\cdot\beta)]$ and $Y = i\beta^i i\frac{\partial}{\partial\alpha^i}$:

$$\begin{aligned}
&[X,Y] = \\
&= -\left[A_i a^i + C(c+\alpha\cdot a - \tfrac{1}{2}a\cdot\beta), \beta^i i\tfrac{\partial}{\partial\alpha^i}\right] \\
&= -[A_i a^i + C(c+\alpha\cdot a - \tfrac{1}{2}a\cdot\beta)]\beta^i i\tfrac{\partial}{\partial\alpha^i} + \beta^i i\tfrac{\partial}{\partial\alpha^i}[A_i a^i + C(c+\alpha\cdot a - \tfrac{1}{2}a\cdot\beta)] \\
&= i\beta^i C a_i.
\end{aligned}$$

And so 2.51 becomes:

$$\begin{aligned}
\tilde{\psi}(\alpha^i) &= e^{i[A_i a^i + C(c+\alpha\cdot a - \frac{1}{2}a\cdot\beta) + \beta^i i\frac{\partial}{\partial\alpha^i} + \frac{1}{2}\beta^i C a_i]}\psi(\alpha^i) \\
&= e^{i[a_i(A^i + C\alpha^i) + Cc + i\beta^i\frac{\partial}{\partial\alpha^i}]}\psi(\alpha^i) \qquad (2.53)\\
&= e^{i(a^i P_i + cI + \beta^i X_i)}\psi(\alpha^i), \qquad (2.54)
\end{aligned}$$

where:

$$\begin{aligned}
\langle\alpha|\, P_i\, |\psi\rangle &:= A_i + C\alpha_i\psi(\alpha) \\
\langle\alpha|\, I\, |\psi\rangle &:= C\psi(\alpha) \\
\langle\alpha|\, X_i\, |\psi\rangle &:= i\frac{\partial}{\partial\alpha^i}\psi(\alpha).
\end{aligned}$$

Without loss of generality, A_i can be taken to 0 and C to the identity. Then:

$$\langle\alpha|\, P_i\, |\psi\rangle := \alpha_i\psi(\alpha) \tag{2.55}$$

$$\langle\alpha|\, I\, |\psi\rangle := \psi(\alpha) \tag{2.56}$$

$$\langle\alpha|\, X_i\, |\psi\rangle := i\frac{\partial}{\partial\alpha^i}\psi(\alpha). \tag{2.57}$$

These operators satisfy the requisite commutation relations of Equation 2.48.

Note that $\mathcal{T}_1(n)$ was initially chosen to be the homogeneous subgroup; this is the "α-representation". Alternatively, $\mathcal{T}_2(n)$ could have been chosen as the homogeneous subgroup, with the normal subgroup therefore being $\mathcal{T}_1(n) \otimes \mathcal{T}(1)$. With this choice, the unitary irreducible representations of the Heisenberg group in the "a-representation" would be:

$$\begin{aligned}
(\tau(g)\psi)(a^i) &= e^{iX_i\alpha^i + iI(c - a\cdot\alpha + \frac{1}{2}\alpha\cdot b)}\psi(a^i - b^i) \\
&= e^{\alpha^i a_i + c - ib^i\frac{\partial}{\partial a^i}}\psi(a) \\
&= e^{i(b^i P_i + cI + \alpha^i X_i)}\psi(a^i), \qquad (2.58)
\end{aligned}$$

[5] See, for example, Chapter 3.1 of [37].

where:

$$\begin{aligned}\langle a|\, P_i\, |\psi\rangle &:= -i\frac{\partial}{\partial a^i}\psi(a)\\ \langle a|\, I\, |\psi\rangle &:= \psi(a)\\ \langle a|\, X_i\, |\psi\rangle &:= a\psi(a).\end{aligned}$$

The Heisenberg group is a central extension of $\mathcal{T}(n)$ which is the subgroup described by the classical case of Section 2.6.2.1. The central charge is given by the partial derivative of the general group element with respect to the parameter c. This is described in more detail in Chapter 7. The Hilbert space in the quantum case is given by:

$$\mathtt{H}^{\tau} = \mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C}, \mu). \tag{2.59}$$

2.6.3 Dimensional Considerations

The arguments set forth in Section 2.6 can be modified to make explicit the connection to space, time, energy and momentum coordinates. With $n = 1$, a^i and α^i can be replaced by, respectively, momentum coordinates $p^\mu = (\frac{e}{\lambda_e}, \frac{p}{\lambda_p})^T$ and position coordinates $x^\mu = (\frac{t}{\lambda_t}, \frac{x}{\lambda_x})^T$, where μ in this case is only equal to 0 or 1, $x_\mu = \eta_{\mu\nu}x^\mu$ and $p_\mu = \eta_{\mu\nu}p^\mu$ (covariant vectors will be represented by row vectors and contravariant vectors by column vectors). The group realisation of Equation 2.46 therefore becomes:

$$g(x^\mu, p^\mu, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{t}{\lambda_t} \\ 0 & 1 & 0 & 0 & 0 & \frac{x}{\lambda_x} \\ 0 & 0 & 1 & 0 & 0 & \frac{p}{\lambda_p} \\ 0 & 0 & 0 & 1 & 0 & \frac{e}{\lambda_e} \\ -\frac{e}{\lambda_e} & \frac{p}{\lambda_p} & -\frac{x}{\lambda_x} & \frac{t}{\lambda_t} & 1 & 2\theta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.60}$$

The order of the e and the p in the matrix have been swapped, and the c in Equation 2.46 has been replaced with θ so as to not be confused with the speed of light. The product of two group elements is therefore given by:

$$g(x^\mu, p^\mu, \theta)g(x'^\mu, p'^\mu, \theta') = g(x^\mu + x'^\mu, p^\mu + p'^\mu, \theta + \theta' + \tfrac{1}{2\hbar}(p^\mu x'_\mu - x^\mu p'_\mu)).$$

The dimensional generators of the algebra are given by (compare with Equations 2.47):

$$T := i\hbar\frac{\partial g}{\partial e} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i\hbar}{\lambda_e} \\ -\frac{i\hbar}{\lambda_e} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.61a}$$

$$X := i\hbar\frac{\partial g}{\partial p} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i\hbar}{\lambda_p} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{i\hbar}{\lambda_p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.61b}$$

$$P := i\hbar\frac{\partial g}{\partial x} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i\hbar}{\lambda_x} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i\hbar}{\lambda_x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.61c}$$

$$E := i\hbar\frac{\partial g}{\partial t} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{i\hbar}{\lambda_t} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i\hbar}{\lambda_t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.61d}$$

$$I := i\frac{\partial g}{\partial \theta} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.61e}$$

The representations of the generators are artificially multiplied by i in order to make the generators Hermitian (cf. Section 2.1.3.1). Multiplying the space-time and momentum-energy generators by $\hbar$ ensures that the generators have the correct dimension; for example the dimensions of X are $\frac{\hbar}{\lambda_p} = \lambda_x$. A general element of the algebra $\mathfrak{Z}$ is therefore:

$$d\mathfrak{Z} = \frac{1}{\hbar}(deT + dpX + dxP + dtE) + \theta I. \tag{2.62}$$

The commutation relations can be calculated directly from these matrix representations, and they are:

$$[E, T] = [X, P] = i\hbar I; \tag{2.63}$$

all other commutators vanish. Equation 2.63 can be written more succinctly as:

$$[X^\mu, P_\nu] = i\eta^\mu{}_\nu I \tag{2.64}$$

The Lie algebra generators can be defined in terms of operators in a position representation or a momentum representation. In the position representation, the

operators are defined to be:

$$\begin{aligned}
\langle x^\mu | \, T \, | \psi \rangle &= t\psi(x^\mu) \\
\langle x^\mu | \, X \, | \psi \rangle &= x\psi(x^\mu) \\
\langle x^\mu | \, P \, | \psi \rangle &= -i\hbar \frac{\partial}{\partial x}\psi(x^\mu) \\
\langle x^\mu | \, E \, | \psi \rangle &= i\hbar \frac{\partial}{\partial t}\psi(x^\mu) \\
\langle x^\mu | \, I \, | \psi \rangle &= I\psi(x^\mu),
\end{aligned}$$

and in the momentum representation the operators are defined to be:

$$\begin{aligned}
\langle p^\mu | \, T \, | \psi \rangle &= i\hbar \frac{\partial}{\partial e}\psi(p^\mu) \\
\langle p^\mu | \, X \, | \psi \rangle &= i\hbar \frac{\partial}{\partial p}\psi(p^\mu) \\
\langle p^\mu | \, P \, | \psi \rangle &= p\psi(p^\mu) \\
\langle p^\mu | \, E \, | \psi \rangle &= e\psi(p^\mu) \\
\langle p^\mu | \, I \, | \psi \rangle &= I\psi(p^\mu).
\end{aligned}$$

The Heisenberg group realisation in Equation 2.60 can be modified by multiplying the first row and dividing the first column by λ_t; multiplying the second row and dividing the second column by λ_x; multiplying the third row and dividing the first column by λ_p; and multiplying the fourth row and dividing the fourth column by λ_e. The matrix, multiplied by $\hbar$, therefore becomes:

$$g(x^\mu, p^\mu, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \hbar t \\ 0 & 1 & 0 & 0 & 0 & \hbar x \\ 0 & 0 & 1 & 0 & 0 & \hbar p \\ 0 & 0 & 0 & 1 & 0 & \hbar e \\ -e & p & -x & t & 1 & 2\hbar\theta \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.65}$$

By abuse of notation, this matrix remains labelled by g. The generators in Equations 2.61 (also maintaining their respective labels by abuse of notation) can then be defined in terms of this matrix instead of the matrix in Equation 2.60. They become:

$$T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\hbar \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.66a}$$

$$X := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.66b}$$

$$P := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.66c}$$

$$E := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & i\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.66d}$$

$$I := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2i\hbar \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.66e}$$

2.7 The Lorentz Group

For a spacetime metric $\eta_{\mu\nu} = \eta^{\mu\nu} = diag\{1, -1, -1, ..., -1\}$ of n spatial dimensions and 1 time dimension, a Lorentz transformation $\Lambda^{\mu}{}_{\nu}$ is a transformation that satisfies:

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta} = \eta_{\alpha\beta}. \tag{2.67}$$

The set of all Lorentz transformations of n spatial dimensions is the Lorentz group, $\mathcal{O}(1, n)$.

The Lorentz group is a connected group with four connected components. From Equation 2.67, each Lorentz transformation can have a determinant of ± 1. Restricting the set of transformations to those which have a determinant of $+1$ gives the proper Lorentz group, $\mathcal{SO}(1, n)$. The remaining transformations are called the "improper Lorentz transformations".

From the $\alpha = \beta = 0$ component of Equation 2.67,

$$\begin{aligned} \eta_{00} &= \eta_{\mu\nu}\Lambda^{\mu}{}_{0}\Lambda^{\nu}{}_{0} \\ \Rightarrow 1 &= \eta_{00}\Lambda^{0}{}_{0}\Lambda^{0}{}_{0} + \eta_{ii}\Lambda^{i}{}_{0}\Lambda^{i}{}_{0} \\ &= (\Lambda^{0}{}_{0})^2 - \sum_i (\Lambda^{i}{}_{0})^2. \end{aligned}$$

From here it can be seen that either $\Lambda^{0}{}_{0} \geq 1$, or $\Lambda^{0}{}_{0} \leq -1$. For the transformations where $\Lambda^{0}{}_{0} > 1$, the direction of time is preserved. These transformations are called orthochronous transformations. The set of all orthochronous Lorentz transformations is the orthochronous Lorentz group, $\mathcal{O}^{+}(1, n)$. Having both restrictions gives the proper orthochronous Lorentz group, $\mathcal{SO}^{+}(1, n)$, which is the

×	I	P	T	PT
I	I	P	T	PT
P	P	I	PT	T
T	T	PT	I	P
PT	PT	T	P	I

Table 2.1: Klein Group Multiplication Table

identity component of the Lorentz group. None of the other three connected components (improper orthochronous, proper non-orthochronous or improper non-orthochronous) are subgroups of the Lorentz group since these sets do not possess an identity element.

The proper orthochronous Lorentz group is a normal subgroup of the Lorentz group. The quotient $\mathcal{O}(1,n)/\mathcal{SO}^+(1,n)$ is isomorphic to a discrete group consisting of four elements: $\{I, P, T, PT\}$. The elements of this group are space inversion P, time-reversal T and the identity I. The remaining element is a transformation that both reverses time and inverts space. This group, denoted by $\mathbb{Z}_{2,2}$, is called the "Klein four-group". It can be written as the direct product of two cyclic groups of order two: $\mathbb{Z}_{2,2} = \mathbb{Z}_2 \times \mathbb{Z}_2$. These cyclic groups are each isomorphic to the set $\{0,1\}$ under addition modulo 2. Alternatively, they are isomorphic to the set $\{1,-1\}$ under multiplication, and they can be represented by matrices of any dimension where the elements of the group are the identity and the negative of the identity. The elements obey the multiplication rules of Table 2.7. This allows the Lorentz group to be written as a semidirect product group:

$$\mathcal{O}(1,n) = \mathbb{Z}_{2,2} \ltimes \mathcal{SO}^+(1,n).$$

In other words, any Lorentz transformation can be written as the product of a proper orthochronous Lorentz transformation and an element of the Klein four-group. If $n = 3$, then the elements of $\mathbb{Z}_{2,2}$ can be realised by the matrices:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$PT = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that the determinant of the parity transformation matrix is -1, which is always true when there are odd spatial dimensions. If $n = 2$, for example, then the parity transformation matrix would be equivalent to a rotation of 180 degrees. And so parity is only a discrete transformation when there are an odd number of spatial dimensions.

2.7.1 Algebra of the Lorentz Group

A good resource for topics on the Lorentz group and its Lie algebra is Kim and Noz [46].

The algebra of the Lorentz group can be deduced by studying the properties of the group elements near the identity. The identity element for the Lorentz group is $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$, so the transformations in question are given by:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \tag{2.68}$$

where the $\omega^{\mu}{}_{\nu}$ are infinitesimally small. These transformations in 2.68 are Lorentz transformations and so they must satisfy Equation 2.67. Therefore:

$$\begin{aligned}\eta_{\alpha\beta} &= \eta_{\mu\nu}(\delta^{\mu}{}_{\alpha} + \omega^{\mu}{}_{\alpha})(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}) \\ &= (\eta_{\nu\alpha} + \omega_{\nu\alpha})\,(\delta^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta}) \\ &= \eta_{\beta\alpha} + \omega_{\alpha\beta} + \omega_{\beta\alpha} + \mathcal{O}(\omega^2).\end{aligned}$$

Excluding the terms to second order in ω this shows that ω is antisymmetric: $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. In $(1+3)$ dimensions, there are therefore six independent components, which can be seen by writing ω as a matrix:

$$\omega = \begin{pmatrix} \omega^0{}_0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3 \\ \omega^1{}_0 & \omega^1{}_1 & \omega^1{}_2 & \omega^1{}_3 \\ \omega^2{}_0 & \omega^2{}_1 & \omega^2{}_2 & \omega^2{}_3 \\ \omega^3{}_0 & \omega^3{}_1 & \omega^3{}_2 & \omega^3{}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3 \\ -\omega^0{}_1 & 0 & \omega^1{}_2 & \omega^1{}_3 \\ -\omega^0{}_2 & -\omega^1{}_2 & 0 & \omega^2{}_3 \\ -\omega^0{}_3 & -\omega^1{}_3 & -\omega^2{}_3 & 0 \end{pmatrix}.$$

With an appropriate choice of phase, the projective representation of the identity, $U(\Lambda_I)$, is equal to the unit operator 1. Therefore, the representation of a Lorentz transformation infinitesimally close to the identity can be written as:

$$U(1+\omega) = 1 + \frac{1}{2} i\omega_{\mu\nu} J^{\mu\nu} + ...$$

(cf. §2.4 of [78]), where the $J^{\mu\nu}$ are operators and there are higher order terms in ω which can be omitted for the purpose of the following analysis.

In order for the representations U to be unitary, the operators $J^{\mu\nu}$ must be Hermitian. Additionally, since ω is antisymmetric, J too must be antisymmetric. A simple calculation shows that $J^{\mu\nu}$ transforms as a tensor under Lorentz transformations:

$$U(\Lambda) J^{\mu\nu} U^{-1}(\Lambda) = \Lambda_{\alpha}{}^{\mu} \Lambda_{\beta}{}^{\nu} J^{\alpha\beta}. \tag{2.69}$$

These operators $J^{\mu\nu}$ are the generators of the Lie algebra of the Lorentz group. They can be written as:

$$J_{\mu\nu} = -i\left(x_{\mu}\frac{\partial}{\partial x^{\nu}} - x_{\nu}\frac{\partial}{\partial x^{\mu}}\right), \tag{2.70}$$

and they satisfy the following commutation relations:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\rho} J^{\mu\sigma} - \eta^{\nu\sigma} J^{\rho\mu} + \eta^{\mu\sigma} J^{\rho\nu}\right). \tag{2.71}$$

Since there are six independent components of ω, there are also six non-zero independent components of J: J^{01}, J^{02}, J^{03}, J^{12}, J^{13} and J^{23}. The first three operators each define a boost transformation and the last three each define a rotation in space. It is often useful to split the generators into the two categories: boosts and rotations. Under this notation each generator is usually denoted by a single index instead of two, with the rotation generators together comprising the angular momentum 3-vector:

$$\mathbf{J} := \{J^1, J^2, J^3\} := \{J^{23}, J^{31}, J^{12}\}. \tag{2.72}$$

The set of all rotations generated by these operators forms an $\mathcal{SO}(3)$-type subgroup of the Lorentz group. The boost generators together comprise the boost 3-vector:

$$\mathbf{K} := \{K^1, K^2, K^3\} := \{J^{01}, J^{02}, J^{03}\}. \tag{2.73}$$

The set of all boosts generated by these operators does not form a subgroup of the Lorentz group, since two successive boosts does not in general yield a third boost. A general Lorentz transformation is then a combination of a rotation and a boost. Under this notation, the commutation relations of the Lie algebra of the Lorentz group are:

$$[J_i, J_j] = i\epsilon_{ij}{}^k J_k, \qquad [J_i, K_j] = i\epsilon_{ij}{}^k K_k, \qquad [K_i, K_j] = -i\epsilon_{ij}^k J_{k}. \tag{2.74}$$

These relations make evident the facts that the rotations form a subgroup while the boosts do not, as well as demonstrating the angular momentum nature of $\mathbf{J}$.

Equation 2.70 can be modified to give the definitions of $\mathbf{J}$ and $\mathbf{K}$ as follows:

$$J_i = -i\epsilon_i{}^{jk} x_j \frac{\partial}{\partial x^k} \tag{2.75}$$

$$K_i = -i\left[x_0 \frac{\partial}{\partial x^i} - x_i \frac{\partial}{\partial x^0}\right] \tag{2.76}$$

$$= -i\left[t\frac{\partial}{\partial x^i} + x_i \frac{\partial}{\partial t}\right] \tag{2.77}$$

A proper orthochronous Lorentz transformation can be written in terms of these generators via the exponential map:

$$\Lambda = e^{i\left(\theta_i J^i + v_i K^i\right)}, \tag{2.78}$$

for parameters θ_i and v_i, $i = 1, 2, 3$, where the doubled index denotes summation over the three values of i with a Euclidean metric. These six parameters have obvious identifications to the parameters $\omega^i{}_j$ seen in Equation 2.68. These parameters can each be used to define a one-parameter subgroup of the proper orthochronous Lorentz group.

As an example, it is well known that a rotation in three dimensions around the z-axis by an angle of θ_3 of a vector $x^\mu = (x^0, x^1, x^2, x^3)$ can be represented by

the following matrix transformation:

$$R(\theta_3)x^\mu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_3 & -\sin\theta_3 & 0 \\ 0 & \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix}.$$

The set of transformations obtained using this transformation matrix for varying values of θ_3 forms a one-parameter subgroup of $\mathcal{SO}^+(1,3)$. A matrix representation of the angular momentum generator J^3 can thus be defined by differentiating this matrix with respect to the parameter θ_3 at the identity (and multiplying by i in order to ensure the representation is Hermitian, and using the same notation J^3 for the representation as for the abstract generator):

$$J^3 = i \left.\frac{\partial R(\theta_3)}{\partial \theta_3}\right|_{\theta_3=0}.$$

The matrix thus obtained is found below in Equations 2.79 together with the other generators.

In a similar fashion, a boost B along the x-axis by a velocity v_1 of x^μ can be represented by the following matrix transformation:

$$B(v_1)x^\mu = \begin{pmatrix} \cosh v_1 & \sinh v_1 & 0 & 0 \\ \sinh v_1 & \cosh v_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \tilde{x}^0 \\ \tilde{x}^1 \\ \tilde{x}^2 \\ \tilde{x}^3 \end{pmatrix}.$$

A matrix representation of the boost generator K^1 is therefore given by (again using the same notation k^1 for the matrix representation as for the abstract generator):

$$K^1 = i \left.\frac{\partial B(v_1)}{\partial v_1}\right|_{v_1=0}$$

This matrix, along with similar matrix representations for the rest of the rest of the generators of the Lorentz algebra are thereby obtained. They are:

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{2.79a}$$

$$J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.79b}$$

$$K^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{2.79c}$$

Returning to the doubled-index notation of Equation 2.70, a general element $\mathfrak{L}$ of the Lorentz algebra is given by:

$$\begin{aligned}\mathfrak{L} &= \theta_{ij}J^{ij} + v_iK^i \\ &= \theta_{\mu\nu}J^{\mu\nu},\end{aligned}$$

where $\theta_{\mu\nu} = \{v_i, \theta_{ij}\}$ are the group parameters.

2.7.2 Universal Covering Group of the Lorentz Group

It is well known that the full Lorentz group $\mathcal{O}(1,3)$ has no intrinsic central charges (cf. Section 2.2.1). That is, it does not admit an algebraic central extension. But $\mathcal{O}(1,3)$ is not simply connected. Therefore its full central extension $\widehat{\mathcal{O}}(1,3)$ is given by its universal covering group $\overline{\mathcal{O}}(1,3)$ (cf. Section 2.2.2):

$$\begin{aligned}\widehat{\mathcal{O}}(1,3) &= \overline{\mathcal{O}}(1,3) \\ &= \overline{\mathbb{Z}}_{2,2} \ltimes \overline{\mathcal{SO}}^+(1,3).\end{aligned}$$

The group $\mathbb{Z}_{2,2}$ is its own cover, so what remains is to determine the cover of $\mathcal{SO}^+(1,3)$. To do this, it is useful to multiply the vector x^μ by σ_μ; the set of Pauli matrices together with σ_0, which is taken to be the identity matrix in two-dimensions. This produces the 2×2 matrix:

$$X = \sigma_\mu x^\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}. \tag{2.80}$$

This matrix has the agreeable property that the determinant is equal to the square of the vector x^μ:

$$\det(X) = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 \tag{2.81}$$

The matrix X is also Hermitian, and this property is preserved under the following transformation:

$$X \mapsto \tilde{X} = AXA^\dagger, \tag{2.82}$$

for any invertible 2×2 matrix A with complex components. The determinant in Equation 2.81 is preserved so long as the determinant of A is ± 1. The set of 2×2 complex matrices with determinant -1 each has the same transformational effect on X - under the transformation of Equation 2.82 - as a matrix which differs by phase only and has a positive determinant. Therefore, matrices with determinant -1 can be discarded, and the remaining matrices are of the following form:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \text{where} \qquad ad - bc = 1. \tag{2.83}$$

In other words, the matrices A belong to the special linear group of dimension 2 with complex components: $\mathcal{SL}(2,\mathbb{C})$. This group is the cover of the Lorentz group and it is simply connected, which means it is the universal covering group

of $\mathcal{SO}^+(1,3)$. This group, along with the proper orthochronous Lorentz group, has six real independent parameters.

However, $\mathcal{SO}^+(1,3)$ is not isomorphic to $\mathcal{SL}(2,\mathbb{C})$. For every matrix $A \in \mathcal{SL}(2,\mathbb{C})$, there is a corresponding matrix $-A$ which produces the same Lorentz transformation. Therefore, $\mathcal{SO}^+(1,3)$ is actually isomorphic to the projective special linear group in two-dimensions with complex components, $\mathcal{PSL}(2,\mathbb{C})$. This is $\mathcal{SL}(2,\mathbb{C})$ with every matrix A identified with its negative $-A$ (note this is not the inverse of A, since the group operation is matrix multiplication and not matrix addition). The projective special linear group is the quotient group of $\mathcal{SL}(2,\mathbb{C})$ with its centre, which is $\mathbb{Z}_2 \cong \{I, -I\}$. In other words:

$$\mathcal{SO}(1,3) \cong \mathcal{PSL}(2,\mathbb{C}) = \mathcal{SL}(2,\mathbb{C})/\mathbb{Z}_2,$$

and

$$\mathcal{O}(1,3) \cong \mathbb{Z}_{2,2} \ltimes \mathcal{SL}(2,\mathbb{C})/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ is the first homotopy group of $\mathcal{SO}^+(1,3)$.

Finally, the (topological) central extension of the Lorentz group is therefore given by:

$$\widehat{\mathcal{O}}(1,3) = \mathbb{Z}_{2,2} \ltimes \mathcal{SL}(2,\mathbb{C}).$$

2.8 The Poincaré Group

Often in the literature the terms "Inhomogeneous Lorentz Group" and "Poincaré Group" are used interchangeably. In the present work however, a distinction will be made between these two groups. The *inhomogeneous Lorentz group* is the semidirect product of the Lorentz group with the group of translations:

$$\mathcal{IO}(1,n) = \mathcal{O}(1,n) \ltimes \mathcal{T}(n+1).$$

The *Poincaré group* is then a central extension of the inhomogeneous Lorentz group, denoted by $\mathcal{P}(1,n)$. The inhomogeneous Lorentz group does not admit an algebraic central extension as shown by [78], and so the central extension is given by the universal covering group of the inhomogeneous Lorentz group. This is:

$$\begin{aligned}\mathcal{P}(1,n) &= \overline{\mathcal{IO}}(1,n) \\ &= \overline{\mathcal{O}}(1,n) \ltimes \overline{\mathcal{T}}(n+1) \\ &= \mathbb{Z}_{2,2} \ltimes \mathcal{SL}(2,\mathbb{C}) \ltimes \mathcal{T}(4)\end{aligned} \tag{2.84}$$

in (1+3) dimensions. This will be called the "extended Poincaré group", as opposed to the proper orthochronous Poincaré group. This chapter will use the term "Poincaré group" to mean either this group or the group $\mathcal{SL}(2,\mathbb{C}) \ltimes \mathcal{T}(4)$, or the inhomogeneous Lorentz group, unless a distinction between these various groups needs to be made.

For elements $\Lambda \equiv k \in \mathcal{O}(1,n)$, $a \in \mathcal{T}(n+1)$, the elements of the semidirect product group are written in the usual way as ordered pairs: (Λ, a). The group product is given by:

$$(\Lambda_1, a_1)(\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2),$$

the inverse is given by:

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a),$$

and the identity element is $(I, 0)$. The elements g of the group can be represented by (non-unitary) 2×2 block matrices:

$$g(\Lambda, a) = \begin{pmatrix} \Lambda & a \\ 0 & 1 \end{pmatrix},$$

or corresponding 5×5 matrices:

$$g(\Lambda, a) = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 & a^0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 & a^1 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 & a^2 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.85}$$

The subgroups $\mathcal{K} \cong \mathcal{O}(1,n)$ and $\mathcal{N} \cong \mathcal{T}(n+1)$ have obvious related representations as 5×5 matrices (using a slight abuse of notation where the representations are denoted by the same symbol as the abstract subgroup elements):

$$k(\Lambda, 0) = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 & 0 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 & 0 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 & 0 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad a(I_n, a) = \begin{pmatrix} 1 & 0 & 0 & 0 & a^0 \\ 0 & 1 & 0 & 0 & a^1 \\ 0 & 0 & 1 & 0 & a^2 \\ 0 & 0 & 0 & 1 & a^3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For pure translations $(I, a_2) \in \mathcal{T}(n+1)$,

$$\begin{aligned} gag^{-1} &= (\Lambda, a_1)(I, a_2)(\Lambda^{-1}, -\Lambda^{-1}a_1) \\ &= (\Lambda, a_1 + \Lambda a_2)(\Lambda^{-1}, -\Lambda^{-1}a_1) \\ &= (I, \Lambda a_2) \\ &\in \mathcal{T}(n+1), \end{aligned}$$

and so $\mathcal{T}(n+1)$ is a normal subgroup. A similar calculation shows that $\mathcal{O}(1,n)$ is not a normal subgroup, so this is not a direct product. As suggested by the name of the group, the Lorentz group $\mathcal{O}(1,n)$ is indeed the *homogeneous* subgroup in the semidirect product.

2.8.1 Poincaré Algebra

The algebra in $(1+3)$ dimensions of the Poincaré group has the same six generators as the Lorentz group, defined in Section 2.7.1, together with another four generators arising from the presence of the translation normal subgroup. The Lorentz generators can be represented by the 4×4 matrices defined in Equations 2.79a-2.79c embedded into 5×5 matrices. These matrices will be detailed below in Equations 2.88-2.90.

The remaining four generators can be calculated directly by taking the partial derivative of the Poincaré transformation matrix in Equation 2.85 - which is a non-unitary representation of a general Poincaré group element - with respect to each element a^μ of the translation subgroup and multiplying by i to maintain Hermiticity (they would otherwise be skew-Hermitian):

$$P^\mu = -i \left. \frac{\partial g(\Lambda, a)}{\partial a^\mu} \right|_{a^\mu = 0} .$$

As will be seen later, the P^0 generator is actually the energy operator - the Hamiltonian - and so it will be relabeled by H. Therefore, the generators of the Poincaré group can be represented by the following matrices:

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad P^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.86}$$

$$P^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad P^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.87}$$

$$J^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad J^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.88}$$

$$J^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^1 = \begin{pmatrix} 0 & i & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.89}$$

$$K^2 = \begin{pmatrix} 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^3 = \begin{pmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.90}$$

These generators have the following commutation relations:

$$[J^i, J^j] = i\epsilon^{ij}{}_k J^k \qquad [J^i, K^j] = i\epsilon^{ij}{}_k K^k \tag{2.91a}$$
$$[K^i, K^j] = -i\epsilon^{ij}{}_k J^k \qquad [J^i, P^j] = i\epsilon^{ij}{}_k P^k \tag{2.91b}$$
$$[K^i, P^j] = i\delta_{ij} H \qquad [K^i, H] = iP^i \tag{2.91c}$$
$$[J^i, H] = [P^i, H] = [H, H] = 0. \tag{2.91d}$$

Just like the Lorentz group, these commutation relations show that the generators $\mathbf{J} = \{J^1, J^2, J^3\}$ form a subgroup. They also commute with the energy, H. Additionally, the 3-vector $\mathbf{P} := \{P^1, P^2, P^3\}$ - the momentum operators - also commute with the energy operator (and they obviously form a subgroup). If the J^i and K^i operators are combined as they were initially in Section 2.7.1 into the single rank-2 tensor $J^{\mu\nu}$, then the commutation relations become:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\nu\sigma}J^{\rho\mu} + \eta^{\mu\sigma}J^{\rho\nu}\right) \tag{2.92}$$
$$[J^{\mu\nu}, P^{\rho}] = i\left(\eta^{\mu\rho}P^{\nu} - \eta^{\nu\rho}P^{\mu}\right), \tag{2.93}$$

and, of course

$$[P^{\mu}, P^{\nu}] = 0. \tag{2.94}$$

The preceding discussion is more properly a discussion of the Lie algebra of the inhomogeneous Lorentz group rather than the Poincaré group. However, there is no algebraic central extension of the Lie algebra of the inhomogeneous group [59, 78], and therefore the Lie algebra of the Poincaré group is isomorphic to the Lie algebra of the inhomogeneous Lorentz group with commutation relations given in Equations 2.91. A general element $\mathfrak{P}$ of this algebra is given by:

$$\begin{aligned}\mathfrak{P} &= \theta_{ij}J^{ij} + v_i K^i + \tfrac{1}{\hbar}(x_i P^i + tH) \\ &= \theta_{\mu\nu}J^{\mu\nu} + \tfrac{1}{\hbar}x_\mu P^\mu,\end{aligned}$$

where the factor of $\hbar$ is introduced by convention.

2.8.2 Representations of the Poincaré Group

The normal subgroup $\mathcal{T}(4)$ is abelian, so the fixed-point condition that defines the little groups is given by Equation 2.36 and the corresponding little groups are defined by Equation 2.37. As the normal subgroup is a translation group, the representations ξ are given by Equation 2.44 for elements a^μ, $\mu = 0, 1, 2, ..., n$ (note that a^0 is the time component). So for states $|p\rangle$ labelled by the four-vector $p \equiv p^\mu$ in the Hilbert space $\mathtt{H}^\xi$:

$$\xi(a^\mu)\,|p\rangle = e^{ip_\mu a^\mu}\,|p\rangle\,. \tag{2.95}$$

If the Hamiltonian of the system is translationally invariant then the Lie algebra operators P^i, $i = 1, 2, 3$ are conserved quantities [26]. These operators have

corresponding eigenvalues which are the character labels:

$$\begin{aligned} P_i \left|p\right\rangle &= -i \left.\frac{\partial \xi(a^i)}{\partial a^i}\right|_{a^i=0} \left|p\right\rangle \\ &= -i \left.\frac{\partial}{\partial a^i} e^{ip_i a^i}\right|_{a^i=0} \left|p\right\rangle \\ &= p_i \left|p\right\rangle . \end{aligned} \tag{2.96}$$

Similarly, the Hamiltonian operator P_0 or H has eigenvalue given by e:

$$H \left|p\right\rangle = e \left|p\right\rangle , \tag{2.97}$$

Note that, due to the choice of metric, the contravariant version of Equation 2.97 has opposite sign whereas the sign in Equation 2.96 remains negative. The explicit inclusion of a factor of $\hbar$ then allows the identification of the P_i operators with momentum, so that:

$$P_i \left|p\right\rangle = \frac{\tilde{p}_i}{\hbar} \left|p\right\rangle ,$$

where $\tilde{p}_i := \hbar p_i$. Similarly, for $\tilde{e} := \hbar c e$:

$$H \left|p\right\rangle = \frac{\tilde{e}}{\hbar c} \left|p\right\rangle .$$

Elements $k \equiv \Lambda$ of the homogeneous subgroup act on the representations of the normal subgroup in the following manner:

$$\begin{aligned} k \cdot \xi &= \xi(kak^{-1}) \\ &= \xi((\Lambda, 0)(I, a^\mu)(-\Lambda, 0)) \\ &= \xi((\Lambda, \Lambda a^\mu)(-\Lambda, 0)) \\ &= \xi(I, \Lambda a^\mu) \\ &= e^{p_\mu (\Lambda \cdot a)^\mu} . \end{aligned}$$

Equation 2.36 is therefore satisfied whenever:

$$\begin{aligned} \xi(I, \Lambda a^i) &= \xi(I, a^\mu) \\ \Rightarrow p_\mu (\Lambda \cdot a)^\mu &= p_\mu a^\mu \\ \Rightarrow p_\nu \Lambda^\nu{}_\mu a^\mu &= p_\mu a^\mu \\ \Rightarrow p_\nu \Lambda^\nu{}_\mu &= p_\mu . \end{aligned}$$

Stated another way,

$$\Lambda^\mu{}_\nu p^\nu = p^\mu$$

In matrix form this is given by:

$$\begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^0{}_1 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^0{}_2 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^0{}_3 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} = \begin{pmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} . \tag{2.98}$$

Following the method of Wigner [81], there are four orbits, two of which have two sub-orbits each. The little groups can then be studied by choosing suitable representative vectors p_μ. The orbits are defined by solutions to the "mass-shell" condition:

$$p_\mu p^\mu \equiv p_0 p^0 - p_i p^i = M^2. \tag{2.99}$$

These orbits are:

- Timelike, where $p_\mu p^\mu > 0$ and so M^2 is positive. There are two sub-orbits, depending upon the sign of p_0. This orbit is discussed in Section 2.8.3.1;
- Spacelike, where $p_\mu p^\mu < 0$ and so M^2 is negative. This orbit is not discussed further as it is non-physical.
- Massless or lightlike, where $p_\mu p^\mu = 0$ and so M is also equal to 0. There are two sub-orbits, again depending upon the sign of p_0. This orbit is discussed in Section 2.8.3.2; and
- Null, where all components of the vector $p_\mu = 0$. This orbit is the trivial Lorentz subgroup of the Poincaré group and is not discussed further.

2.8.3 Representations of the Little Groups

2.8.3.1 Timelike Case

The timelike orbit is the orbit defined by $p_\mu p^\mu > 0$. A representative vector is $\widehat{p}^\mu = (\pm M, 0, 0, 0)^T$, for $n = 3$ spatial dimensions, which has a length of M. Then the orbit $\mathcal{SO}(1,3) \cdot \widehat{p}^\mu$ is the set of images obtained through proper orthochronous Lorentz transformations acting on this representative vector. There are sub-orbits defined by the sign of $\widehat{p}^0$; if $\widehat{p}^0$ is positive then the vector and the orbit are said to be *future-pointing*. If, on the other hand, $\widehat{p}^0$ is negative then the orbit and the vector are said to be *past-pointing*. The little group for both sub-orbits are the same; it is the set of all Lorentz transformations that leave this representative vector invariant. The length M of the vector can be used to label the representations of this orbit.

If $\widehat{p}^\mu = (\pm M, 0, 0, 0)^T$, then Equation 2.98 shows that ${\Lambda^0}_i = {\Lambda^i}_0 = 0$, for $i = 1, 2, 3$, and ${\Lambda^0}_0 = 1$. The little group then, is the maximal set of remaining elements. In matrix form, it is the set of all proper orthochronous Lorentz matrices $\Lambda^\xi \in \mathcal{SO}(1,3)$ of the following form:

$$\Lambda^\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & {\Lambda^1}_1 & {\Lambda^1}_2 & {\Lambda^1}_3 \\ 0 & {\Lambda^2}_1 & {\Lambda^2}_2 & {\Lambda^2}_3 \\ 0 & {\Lambda^3}_1 & {\Lambda^3}_2 & {\Lambda^3}_3 \end{pmatrix}$$

This is a 4×4 matrix realisation of the group $\mathcal{SO}(3)$ - the group of all rotations in 3-dimensions.

The stabiliser group is:

$$\mathcal{G}^{\xi} = \mathcal{SO}(3) \ltimes \mathcal{T}(4),$$

and the representations of the elements of the stabiliser group are given by Equation 2.38, as:

$$\tau^{\xi}(g) = \sigma(R) \otimes \xi(a), \tag{2.100}$$

for $g \in \mathcal{G}^{\xi}$ and where $R \equiv \Lambda^{\xi}$ are elements of $\mathcal{SO}(3)$ - rotations. Define $\mathbb{X}$ to be the quotient space of $\mathcal{G}/\mathcal{G}^{\xi}$. That is, $\mathbb{X}$ is the three dimensional hyperbolic space $\mathbb{H}^3$. Then Θ is the coset representative function that takes elements of $x \in \mathbb{H}^3$ to $\mathcal{P}(1,3)$:

$$\Theta : x \in \mathbb{H}^3 \to \mathcal{P}(1,3).$$

These cosets will be labelled by a subscript B, which indicates a pure boost transformation. Then:

$$x_B = g(B,0)\mathcal{G}^{\xi}.$$

The function Θ takes this coset to an element of the group $\mathcal{P}(1,3)$ and so:

$$\Theta(x_B) = g(B,0).$$

To determine all the terms in Equation 2.41 requires the result of the product of a group element (actually, the inverse of that element) and the coset. By separating the Lorentz transformation in the group element into a pure boost and a pure rotation, this yields:

$$\begin{aligned} g x_{B'} &= (\Lambda, a)(B', 0)\mathcal{G}^{\xi} \\ &= (BB', 0)(R, a)\mathcal{G}^{\xi} \\ &= (BB', 0)\mathcal{G}^{\xi}; \\ \therefore \Theta(g^{-1}x) &= x_{B^{-1}B'}. \end{aligned}$$

All the terms in Equation 2.41 can now be easily expressed and the representations of the group can be written in terms of Equation 2.42. Then, for $\Lambda := BR$:

$$\begin{aligned} \langle x_{B'} | \, \tau \, | \psi \rangle &= \langle x_{B^{-1}B'} | \, \tau^{\xi}[g(B'^{-1}, 0)g(\Lambda, a)g(B^{-1}B', 0)] \, | \psi \rangle \\ &= \tau^{\xi}(B'^{-1}BRB^{-1}B', -B'a)\psi(x_{B^{-1}B'}) \\ &= \tau^{\xi}(R^{\circ}, -B'a)\psi(x_{B^{-1}B'}), \end{aligned}$$

where $R^{\circ} = B'^{-1}BRB^{-1}B'$ is a pure rotation. Then, from Equation 2.100:

$$\begin{aligned} \langle x_{B'} | \, \tau \, | \psi \rangle &= \sigma(R^{\circ}) \otimes \xi(-B'a)\psi(x_{B^{-1}B'}) \\ &= \sigma(R^{\circ}) e^{-ic_{\mu}(B'a)^{\mu}} \psi(x_{B^{-1}B'}), \end{aligned}$$

where c_μ are character labels. The representations are guaranteed by the Mackey induction method to be irreducible, and they will be unitary so long as the representations $\sigma(R^\circ)$ are unitary. The unitary irreducible representations of the group $\mathcal{SO}(3)$ are well known.

So far, the representations discussed have been the representations of the previously defined Inhomogeneous Lorentz Group. To change to representations of the (proper orthochronous) Poincaré group, all that changes is the representations σ of the little group. It is well known that the universal covering group of $\mathcal{SO}(3)$ is $\mathcal{SU}(2)$. The extension, therefore, to the Poincaré group simply involves the addition of space inversion, time reversal and the combination of space inversion and time reversal - in other words, the operations summarised in Table 2.7.

The Hilbert space of the stabiliser group is given by Equation 2.39 as $\mathtt{H}^{\sigma^\xi} \otimes \mathbb{C}$. The Hilbert space of the whole group is therefore given by:

$$\begin{aligned}\mathtt{H}^\tau &\cong \mathtt{H}^{\sigma^\xi} \otimes \mathbb{C} \otimes \mathcal{L}^2(\mathbb{X}, \mathbb{C}, \mu) \\ &= \mathcal{L}^2(\mathbb{H}^3, \mathtt{H}^{\sigma^\xi}, \mu),\end{aligned}$$

where μ is a measure on $\mathbb{H}^3$. The Hilbert space $\mathbb{H}^{\sigma^\xi}$ is the Hilbert space of the little group - in this case it is the group $\mathcal{SU}(2)$.

As any standard quantum mechanics text book will describe[6], the timelike states can be expressed as simultaneous eigenstates of the angular momentum operator in a given direction - by convention the z or 3 direction - as well as the square of the vector $\mathbf{J} = (J_1, J_2, J_3)$ of angular momentum operators:

$$\mathbf{J}^2 = {J_1}^2 + {J_2}^2 + {J_3}^2.$$

This operator commutes with each component of angular momentum, and has eigenvalues such that:

$$\mathbf{J}^2 \left|\widehat{p}\right\rangle = s(s+1) \left|\widehat{p}\right\rangle .$$

The number s is called the "spin quantum number", and it takes the values: $s = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ The half-integral values arise due to the fact that the universal covering group $\mathcal{SU}(2)$ of the timelike little group is the double cover of $\mathcal{SO}(3)$. The J_3 operator has eigenvalues m_s:

$$J_3 \left|\widehat{p}\right\rangle = m_s \left|\widehat{p}\right\rangle .$$

This quantum number is called the "secondary spin quantum number" and takes the values: $m_s = -s, -s+1, ..., s-1, s$. Since the timelike states can be simultaneously diagonalised in both of these operators, the states can therefore be labelled by the quantum numbers s and m_s. The states still transform under translations like Equation 2.95, and so the label $\widehat{p}$ will be retained. Therefore the two quantum numbers along with the momentum $\widehat{p}$ can be used to label the timelike representations as the $2s+1$ basis vectors $|\widehat{p}; sm_s\rangle$. The corresponding Poincaré unitary irreducible representations for the timelike case can be denoted

[6] See, for example, [16]

by $\tau^{(M,s)}$ where s is the spin quantum number and M is the (positive) length of the momentum four-vector. Since there are an infinite number of p for any length M, the representations are infinite dimensional [26].

In timelike representations the value of $p_\mu p^\mu$ is given by M^2. By replacing M with a quantity that has dimensions of mass:

$$\tilde{M} := \frac{\hbar}{c} M,$$

it follows from the form of $P^\mu P_\mu = {P_0}^2 - \sum_i {P_i}^2$, using the respective Lie algebra eigenvalues, that:

$$\begin{aligned} \frac{\tilde{M}^2 c^2}{\hbar^2} &= \frac{\tilde{e}^2}{\hbar^2 c^2} - \sum_i \frac{\tilde{p}_i^2}{\hbar^2} \\ \Rightarrow \tilde{e}^2 &= \tilde{M}^2 c^4 + c^2 \tilde{\mathbf{p}}^2, \end{aligned}$$

which is the relativistic energy-momentum equation.

2.8.3.2 Massless Case

The massless orbit is the orbit defined by $p_\mu p^\mu = 0$, where $p^\mu \neq (0,0,0,0)^T$. A representative vector for this orbit is $\widehat{p}^\mu = (\pm M, 0, 0, \pm M)^T$ where both the signs must be the same. For the forward-pointing sub-orbit both components must be positive. Following Wigner [81] (and cf. [79]), the determination of the little group for the massless case is best done by representing the Lorentz transformations as in Section 2.7.2. This has $\mathcal{SL}(2,\mathbb{C})$ matrices A obeying the transformation rule of Equation 2.82 with matrices X defined in Equation 2.80. Substituting the forward-pointing representative vector $\widehat{p}^\mu$ into Equation 2.82 gives:

$$\begin{aligned} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2M & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} 2M & 0 \\ 0 & 0 \end{pmatrix} \\ \Rightarrow \quad & \begin{pmatrix} 2Maa^* & 0 \\ 2Mca^* & 0 \end{pmatrix} = \begin{pmatrix} 2M & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

From this, it follows that:

$$\begin{aligned} c &= 0, \\ |a|^2 &= 1. \end{aligned}$$

Recalling that $ad - bc = 1$, take $a = e^{-i\phi/2}$, and $d = e^{i\phi/2}$, as $ad - bc = 1 \Rightarrow d = 1/a$. Still following the argument of [81], the most general element Λ^ξ of the little group is given by:

$$\Lambda^\xi = \begin{pmatrix} e^{-i\phi/2} & (x+iy)e^{i\phi/2} \\ 0 & e^{i\phi/2} \end{pmatrix}.$$

The element b of the general form of the $\mathcal{SL}(2,\mathbb{C})$ matrix is chosen in this manner so that the little group matrices Λ^ξ are easily decomposed into two subgroups of

matrices, $B(x,y)R(\theta)$:

$$B(x,y) = \begin{pmatrix} 1 & x+iy \\ 0 & 1 \end{pmatrix}; R(\phi) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}. \tag{2.101}$$

$R(\theta)$ is a rotation matrix of an angle ϕ around the 3 axis, and $B(x,y)$ is a boost of magnitude x in the 1 direction, and y in the 2 direction. Clearly, these matrices satisfy the relations:

$$\begin{aligned} B(x_1,y_1)B(x_2,y_2) &= B(x_1+x_2, y_1+y_2) \\ R(\phi_1)R(\phi_2) &= R(\phi_1+\phi_2). \end{aligned}$$

This little group is isomorphic to the group $\mathcal{E}(2) = \mathcal{O}(2) \ltimes \mathcal{T}(2)$ - the two dimensional Euclidean group, or inhomogeneous rotation group. The generators of this group can be constructed from the Pauli-Lubanski vector [76], which is defined as:

$$W^\lambda := \tfrac{1}{2}\epsilon^{\lambda\mu\nu\sigma} J_{\mu\nu} P_\sigma. \tag{2.102}$$

In the timelike case the momentum operator P_σ has only two non-zero components (both of which have the value of M for the forward-pointing sub-orbit), and so the generators of $\mathcal{E}(2)$ are:

$$\begin{aligned} W^0 &= W^3 = MJ_{12} = MJ_3 \\ W^1 &= M(J_{20} + J_{23}) = M(-K_2 + J_1) \\ W^2 &= M(J_{01} + J_{31}) = M(K_1 + J_2). \end{aligned}$$

These generators obey the following set of commutation relations:

$$\begin{aligned} \left[J_3, W^1\right] &= iW^2 \\ \left[J_3, W^2\right] &= -iW^1 \\ \left[W^1, W^2\right] &= 0. \end{aligned}$$

A general Lorentz transformation can be broken up into two parts:

$$\Lambda = XL, \tag{2.103}$$

where $L = B(x,y)R(\phi)$ involves only transformations of the kind defined in Equation 2.101 - the little group portion of the Lorentz transformation. The X portion of the Lorentz transformation therefore involves a boost $B(z)$ along the 3 axis and a rotation $R(\theta,\gamma)$ which brings the orientation of the representative vector $\widehat{p}^\mu$ into the direction of a general vector p^μ. This means that any vector p^μ can be written as:

$$\begin{aligned} p^\mu &= R(\theta,\phi)B(z)\widehat{p}^\mu \\ &= X\widehat{p}^\mu. \end{aligned} \tag{2.104}$$

The cosets $x \in \mathcal{G}/\mathcal{G}^\xi$ can now be written as:

$$\begin{aligned} x &= (R(\theta,\gamma)B(z),0)\mathcal{G}^\xi \\ &= (X,0)\mathcal{G}^\xi. \end{aligned}$$

For brevity, these cosets will be denoted x_X. In the notation of Equation 2.103, an inverse group element of the Poincaré group is, in general, $(L^{-1}X^{-1}, -L^{-1}X^{-1}a)$. Therefore:

$$\begin{aligned} g^{-1}x_{X'} &= (L^{-1}X^{-1}, -L^{-1}X^{-1}a)(X',0)\mathcal{G}^\xi \\ &= (X^{-1}X',0)(L^{-1},-L^{-1}a)\mathcal{G}^\xi \\ &= (X^{-1}X',0)\mathcal{G}^\xi \\ &= x_{X^{-1}X'} \end{aligned}$$

and

$$\Theta(g^{-1}x_{X'}) = (X^{-1}X',0).$$

All the terms in Equation 2.41 can now be easily expressed and the representations of the group can be written in terms of Equation 2.42:

$$\begin{aligned} \langle x_{X'}|\,\tau\,|\psi\rangle &= \langle x_{X'}|\,\tau(\Theta^{-1}(x_{X'})g\Theta(g^{-1}x)\,|\psi\rangle \\ &= \langle x_{X^{-1}X'}|\,\tau^\xi\left[(X'^{-1},0)(XL,a)(X^{-1}X',0)\right]|\psi\rangle \\ &= \tau^\xi\left[(X'^{-1}XL,-X'a)(X^{-1}X',0)\right]\psi_{X^{-1}X'} \\ &= \tau^\xi((X'^{-1}XLX^{-1}X',-X'a))\psi_{X^{-1}X'} \\ &= \tau^\xi((L^\circ,-X'a))\psi_{X^{-1}X'}, \end{aligned}$$

where L° is a transformation of the little group. From Equation 2.38:

$$\begin{aligned} \langle x_{X'}|\,\tau\,|\psi\rangle &= \sigma(L^\circ)\otimes\xi(-X'a) \\ &= \sigma(L^\circ)e^{-ic_\mu(X'a)^\mu}\psi_{X^{-1}X'}, \end{aligned}$$

where c_μ are character labels and it should be recalled that X and X' are transformations of the type seen in Equation 2.104.

The Hilbert space of the stabiliser group is given in Equation 2.39 as $\mathtt{H}^{\sigma^\xi}\otimes\mathbb{C}$. The Hilbert space of the whole group in the massless case is therefore:

$$\begin{aligned} \mathtt{H}^\tau &\cong \mathtt{H}^{\sigma^\xi}\otimes\mathbb{C}\otimes\mathcal{L}^2(\mathbb{X},\mathbb{C},\mu) \\ &= \mathcal{L}^2(\mathbb{X},\mathtt{H}^{\sigma^\xi},\mu), \end{aligned}$$

where μ is a measure on $\mathbb{X}$ and $\mathtt{H}^{\sigma^\xi}$ is the Hilbert space of the little group - in this case, the group $\mathcal{E}(2)$.

As in Section 2.8.3.1 for timelike states, the massless states can be expressed as simultaneous eigenstates of the J_3 operator as well as the square of the vector $\mathbf{W} := (W^1, W^2)$:

$$\mathbf{W}^2 = W^{1\,2} + W^{2\,2}. \tag{2.105}$$

This operator commutes with all three generators of the $\mathcal{E}(2)$ algebra. In a basis labelled by the representative vector $|\widehat{p}\rangle$, the $\mathbf{W}^2$ operator has eigenvalues w:

$$\mathbf{W}^2 |\widehat{p}\rangle = w |\widehat{p}\rangle$$

and the J_3 generator has eigenvalues λ:

$$J_3 |\widehat{p}\rangle = \lambda |\widehat{p}\rangle .$$

Therefore massless states can be labelled by these numbers. Retaining the $\widehat{p}$ label in the ket, the basis vectors will be labelled as $|\widehat{p}; w\lambda\rangle$. However, there has never been a particle discovered with a non-zero w and zero M [76]. In the case of massless particles, the λ label, which corresponds to m_s in timelike particles, relates to the helicity of the particle.

2.8.4 Poincaré Group Field Equations

2.8.4.1 Casimir Operators of the Poincaré Algebra

The first Casimir operator of the Poincaré Lie algebra is the momentum operator squared:

$$C_1 := P_\mu P^\mu. \tag{2.106}$$

This operator is invariant under Lorentz transformations since the square of any four-vector is always invariant under Lorentz transformations. It is clearly invariant under space-time translations. In states $|p\rangle$ that transform under translations like Equation 2.95, the eigenvalues of the Lie algebra generators are given in Equations 2.96 and 2.97 as:

$$P_\mu |p\rangle = p_\mu |p\rangle ,$$

where $p_\mu = (e, \mathbf{p})^T$. If the length of the four-vector p_μ is given by M, then $M^2 = -e^2 + \mathbf{p}^2$ and:

$$\begin{aligned} C_1 |p\rangle &:= P_\mu P^\mu |p\rangle \\ &= (-{p_0}^2 + {p_1}^2 + {p_2}^2 + {p_3}^2) |p\rangle \\ &= M^2 |p\rangle . \end{aligned}$$

In other words, the Casimir operator C_1 is defined to be:

$$C_1 = M^2 I, \tag{2.107}$$

where I is the identity operator. In the timelike representation $\tau^{(M,s)}$ the Casimir operator is the mass squared $\frac{c^2}{\hbar^2}\tilde{M}^2$. In the massless representation $\tau^{(0,s)}$ the Casimir operator has a value of zero, although this does not mean that the components of p_μ are all zero (this would correspond instead to the null case).

The second Casimir operator is constructed from the Pauli-Lubanski operator W^μ. This operator is defined in Equation 2.102 and has the following properties [76]:

$$\begin{aligned} W^\mu P_\mu &= 0 \\ \left[W^\lambda, P^\mu\right] &= 0 \\ \left[W^\lambda, J^{\mu\nu}\right] &= i(W^\mu \eta^{\lambda\nu} - W^\nu \eta^{\mu\lambda}) \\ \left[W^\lambda, W^\sigma\right] &= \tfrac{1}{2}\epsilon^{\mu\nu\lambda\sigma} J_{\mu\nu} P_\lambda P_\sigma. \end{aligned}$$

The second Casimir operator is then defined to be:

$$C_2 := W_\mu W^\mu \tag{2.108}$$

In the timelike representation $\tau^{(M,s)}$ and with a basis state $|p; sm_s\rangle$ where $p = (M, 0, 0, 0)^T$ the momentum Lie algebra generators do the following:

$$P_i |p; sm_s\rangle = 0, \qquad P_0 |p; sm_s\rangle = M |p; sm_s\rangle .$$

Therefore, the components of the Pauli-Lubanski operator do the following:

$$\begin{aligned} W_0 |p; sm_s\rangle &= 0, \\ W_1 |p; sm_s\rangle &= (\tfrac{1}{2}\epsilon^{1230} J_{23} P_0 + \tfrac{1}{2}\epsilon^{1320} J_{32} P_0) |p; sm_s\rangle \\ &= M J_1 |p; sm_s\rangle . \end{aligned}$$

Similarly, $W_2 |p; sm_s\rangle = M J_2 |p; sm_s\rangle$ and $W_3 |p; sm_s\rangle = M J_3 |p; sm_s\rangle$ (cf. Section 2.7.1 for the notation of $J_{\mu\nu}$ and J_i). Therefore, for timelike representations, the value of C_2 is given by:

$$\begin{aligned} C_2 |p; sm_s\rangle :&= W_\mu W^\mu |p; sm_s\rangle \\ &= M^2 \mathbf{J}^2 |p; sm_s\rangle \\ &= M^2 s(s+1) |p; sm_s\rangle , \end{aligned}$$

where the fact that $\mathbf{J}^2$ is the Casimir operator of $\mathcal{SO}(3)$ with eigenvalue $s(s+1)$ has been used. This allows the second Casimir operator of the Poincaré group to be defined as:

$$C_2 := M^2 s(s+1) I$$

for timelike particles. The second Casimir operator in the massless case, however, has already been defined in Equation 2.105. For physical massless particles, this Casimir operator has eigenvalue $w = 0$.

2.8.4.2 Measure

An arbitrary state vector $|\psi\rangle$ can be expanded in terms of the basis vectors $|p\lambda\rangle$ as follows:

$$|\psi\rangle = \sum_\lambda \int \psi^\lambda(p) |p\lambda\rangle \, \tilde{dp},$$

where $\tilde{dp}$ is the integration measure. This measure is given by Tung (in [76]) as:

$$\begin{aligned}\tilde{dp} &= \frac{1}{N}\frac{d^3\mathbf{p}}{2p^0} \\ &= \frac{1}{(2\pi)^3}\frac{d^3\mathbf{p}}{2p^0},\end{aligned} \tag{2.109}$$

with $N = (2\pi)^3$ by convention. The value of p^0 is given by the "mass-shell" condition - Equation 2.99 - and the first Casimir. This condition states that:

$$M^2 - p_\mu p^\mu = M^2 + \mathbf{p}^2 - (p^0)^2 = 0,$$

so that:

$$p^0 = \pm\sqrt{\mathbf{p}^2 + M^2}.$$

Only the positive root applies for forward-pointing representations. Therefore Equation 2.109 can be modified to:

$$\tilde{dp} = \frac{1}{(2\pi)^3}\frac{d^3\mathbf{p}}{2(\mathbf{p}^2 + M^2)}. \tag{2.110}$$

2.8.4.3 The Klein-Gordon Equation

The Klein-Gordon Equation can be derived directly from the eigenvalue equation of the first Casimir operator for timelike unitary irreducible representations. This equation is:

$$\begin{aligned} P_\mu P^\mu \left|p; sm_s\right\rangle &= M^2 \left|p; sm_s\right\rangle \\ \Rightarrow (-{P_0}^2 + \mathbf{P}^2 - M^2)\left|p; sm_s\right\rangle &= 0 \\ \Rightarrow (\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \frac{c^2}{\hbar^2}\tilde{m}^2)\left|p; sm_s\right\rangle &= 0.\end{aligned}$$

Similarly to this derivation the Dirac equation, the Weyl equation and Maxwell's equations can all be derived from the eigenvalue equations of the representations of the Casimir operators of the central extension of a symmetry group. However, the other derivations are not quite as simple as the derivation for the Klein-Gordon equation. See [26] for the other derivations.

Chapter 3

Reciprocity and the Quaplectic Group

The theory of reciprocity has undergone many stages in its development. It was first developed by Max Born in the 1930s, who noticed a fundamental symmetry between space and momentum in several different areas of physics. See Section 1.4 for a discussion of the work done by Born and his contemporaries; in particular, Herbert S. Green. Since the death of Born, the popularity of the theory has considerably declined.

A more modern approach has been suggested by Stephen Low, using representation theory of groups to develop a group which is invariant under all reciprocal transformations. This approach and the resulting group, the quaplectic group, will be discussed in Chapter 3.1. The theory suggests a new kind of relativity, called "reciprocal relativity", which is discussed in Section 3.2. In particular, the concepts of time dilation and length contraction in reciprocal relativity are discussed. This section is mostly a review of work done by Low, except for the example of the application of time dilation to a system of constant relative acceleration. The chapter is concluded by looking at how proper acceleration works in a quaplectic framework in Section 3.2.1.

3.1 Low and the Quaplectic Group

The theory of special relativity can be developed from purely group theoretical considerations, as shown in Section 2.8. The Poincaré group is the most general group which keeps invariant transformations between frames of the $n+1$-dimensional line element of space-time. This line element is:

$$ds^2 = -dt^2 + \frac{1}{c^2}dx^2. \tag{3.1}$$

Special relativity defines transformations between frames of observers that are moving with a relative velocity of v. However, the frames must be inertial with respect to one-another. This is a limitation of the theory and it is only partially resolved in general relativity, which bypasses the problem by making non-inertial frames locally inertial. In addition, much of the physics of quantum mechanics

does not arise naturally out of the Poincaré group, and so this must be artificially added to the theory.

These limitations provide the motivation for the search for a different dynamical symmetry group which does not suffer from these limitations. One candidate for this is the quaplectic group, so named by Low as discussed by him in [53, 54, 55, 56, 57, 58, 60, 62]. The quaplectic group is a semidirect product group, with the normal subgroup being the Heisenberg group. This means that Heisenberg's Uncertainty Principle and the canonical commutation relations of quantum mechanics are automatically included in the theory. If the theory is to be non-inertial, then it must be able to take into account what happens when two frames are interacting with one another, causing a respective force. Note that this force is properly defined as a relative force between two respective reference frames - it is not defined relative to some kind of "absolute noninertial rest frame", which is not assumed to exist in the theory (neither is it assumed not to exist). The relative force between respective reference frames will be denoted by parameters f^i, and the reference frames will be said to "move at a relative force f^i". Additionally, the quaplectic group allows for a relative power between two respective reference frames. This will be denoted by r, and the reference frames will be said to be "moving at a relative power r".

To construct a semidirect product group from the Heisenberg group, following Section 2.3, the homogeneous group must be a subgroup of the automorphism group of the normal subgroup. The automorphism group of the Heisenberg group $\mathcal{H}(n+1)$ is the following semidirect product group involving the symplectic group $\mathcal{S}p(2n+2)$ as the homogeneous group:

$$\mathcal{A}ut(\mathcal{H}(n+1)) \cong \mathbb{Z}_2 \otimes \mathcal{D}(1) \otimes \mathcal{S}p(2n+2) \ltimes \mathcal{H}(n+1),$$

where $\mathbb{Z}_2$ is the cyclic group of order 2 and $\mathcal{D}(1)$ is the one parameter abelian group of dilations, $r > 0$. For a Heisenberg element (α^i, a^i, c), a dilation maps the element to $(r\alpha^i, ra^i, r^2c) \in \mathcal{H}(n+1)$ [30]. Low then postulates that the relevant symmetry group should contain the structure of an invariant line element. In other words, the homogeneous group is the intersection of the symplectic group $\mathcal{S}p(2n+2)$ and the pseudo-orthogonal group $\mathcal{O}(2, 2n)$. This yields a pseudo-unitary group:

$$\mathcal{U}(1, n) = \mathcal{S}p(2n+2) \cap \mathcal{O}(2, 2n).$$

Therefore, the quaplectic group is defined as:

$$\mathcal{Q}(1, n) = \mathcal{U}(1, n) \ltimes H(n+1). \tag{3.2}$$

The pseudo-orthogonal metric is:

$$\eta^{MN} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where I_n is simply the identity matrix in n dimensions and the capital Roman indices $M, N = 0, 1, ..., n$ correspond to $2n+2$ dimensional n-phase space. In

1-phase space this metric yields the following line element:

$$ds^2 = -dt^2 + \frac{1}{c^2}dx^2 + \frac{1}{b^2}(dp^2 - \frac{1}{c^2}de^2),$$

the generalisation to 3-phase space or n-phase space being obvious. The constant b (which is unrelated to Born's b in Section 1.4) is a new fundamental constant with dimensions of force. Just as c is the maximum rate of change of position, this new constant b is the maximum rate of change of momentum. The value of the constant is unknown and must be determined by future experiment.

The unitary irreducible representations of the quaplectic group are discussed in Section 3.1.1. The algebra is then discussed in Section 3.1.3 and the field equations in Section 3.1.4. The reciprocal relativity is discussed in Section 3.2.

Any new physics in Section 3.1 is attributable to Low.

3.1.1 Representations of the Quaplectic Group

The unitary irreducible representations of the quaplectic group $\mathcal{Q}(1,n)$ can be derived using the process described in Sections 2.3 and 2.4. The normal subgroup is the Heisenberg group $\mathcal{H}(n+1)$, and the unitary irreducible representations of this group are found in Equation 2.54. The homogeneous subgroup is the pseudo-unitary group $\mathcal{U}(1,n)$.

Elements $k \in \mathcal{U}(1,n)$ act on the unitary dual of the Heisenberg group, which is the set of equivalence classes $[\xi]$ of unitary irreducible representations ξ of the Heisenberg group elements a. Since the Heisenberg group is non-abelian, the little groups of the semidirect product are defined in Equation 2.32 as:

$$\mathcal{K}^\xi = \{k \in \mathcal{U}(1,n) : k \cdot \xi = \rho(k)\xi\rho^{-1}(k)\},$$

where ρ is a projective extension representation of the representations ξ to the stabiliser group $\mathcal{G}^\xi$, and ρ acts on the Hilbert space of the Heisenberg group:

$$\rho(g) : \mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C}, \mu) \to \mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C}, \mu), \qquad \forall g \in \mathcal{Q}p(1,n).$$

The projective extension representations have the property that $\rho|_{\mathcal{H}(n+1)} \cong \xi$.

Wolf shows in [82] that the little group is the whole of the pseudo-unitary group $\mathcal{U}(1,n)$. In other words, the fixed point condition:

$$k \cdot \xi = \rho(k)\xi\rho^{-1}(k)$$

is true for all $k \in \mathcal{U}(1,n)$, and therefore $\mathcal{K}^\xi = \mathcal{U}(1,n)$. This means that the Mackey induction method does not need to be used. The representations of the stabiliser group $\mathcal{K}^\xi \ltimes \mathcal{H}(n+1)$, given by:

$$\tau^\xi = \sigma \otimes \rho \tag{3.3}$$

are therefore the representations of the quaplectic group: $\tau \equiv \tau^\xi$. Additionally, since the little group is the whole homogeneous group, $\mathtt{H}^{\sigma^\xi} = \mathtt{H}^\sigma$ and the Hilbert

space $\mathtt{H}^{\tau^\xi} = \mathtt{H}^\sigma \otimes \mathtt{H}^\xi$ of the stabiliser group is also the Hilbert space of the quaplectic group: $\mathtt{H}^\tau = \mathtt{H}^{\tau^\xi}$. Therefore:

$$\mathtt{H}^\tau = \mathtt{H}^\sigma \otimes \mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C}, \mu).$$

If the Hilbert space of the quaplectic group $\mathtt{H}^\tau$ consists of kets $|\psi\rangle$, the Hilbert space of the homogeneous group $\mathtt{H}^\sigma$ consists of kets $|\phi\rangle$ and the Hilbert space of the projective extension representations $\mathtt{H}^\rho = \mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C}, \mu)$ of the normal subgroup consists of kets $|\theta\rangle$, then, for elements $g = (k, a)$ of the quaplectic group, from Equation 3.3

$$\tau(g)\,|\psi\rangle = \sigma((k,0)\,|\phi\rangle \otimes \rho(g(k,a))\,|\theta\rangle\,;$$

where the fact that $\tau \equiv \tau^\xi$ has been used. Thus the unitary irreducible representations of the quaplectic group are defined in terms of the unitary irreducible representations σ of the group $\mathcal{U}(1,n)$ and the projective extension representations ρ of the normal subgroup $\mathcal{H}(n+1)$. The unitary irreducible representations σ of the pseudo-unitary group $\mathcal{U}(1,n)$ are well-known [70, 67] and the Hilbert space the representations act upon is a countably infinite complex vector space [58]: $\mathbb{H}^\sigma = \mathbb{V}^\infty$. The Hilbert space of the quaplectic group is given by:

$$\mathtt{H}^\tau = \mathbb{V}^\infty \otimes \mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C}),$$

since $\mathcal{L}^2(\mathbb{R}^{n+1}, \mathbb{C})$ is the Hilbert space of the Heisenberg group, from Equation 2.59.

3.1.2 Quaplectic Matrix Realisations in 1-Phase Space

It is useful to define a (non-unitary) matrix realisation of elements of the quaplectic group. For $n = 1$, the elements of the homogeneous group $\mathcal{U}(1,1)$ can be written in a matrix realisation [58][1] as:

$$\Xi(v,f,r,a) = \Xi^\circ(a)\Xi(v,f,r), \tag{3.4}$$

where

$$\Xi(v,f,r) = \gamma_{vfr} \begin{pmatrix} 1 & \frac{v}{c} & \frac{f}{b} & -\frac{r}{bc} \\ \frac{v}{c} & 1 & \frac{r}{bc} & -\frac{f}{b} \\ \frac{f}{b} & -\frac{r}{bc} & 1 & \frac{v}{c} \\ \frac{r}{bc} & -\frac{f}{b} & \frac{v}{c} & 1 \end{pmatrix} \tag{3.5}$$

is a group element of $\mathcal{SU}(1,1)$, and

$$\Xi^\circ(a) = \frac{1}{\sqrt{1 + \left(\frac{a}{bc}\right)^2}} \begin{pmatrix} 1 & 0 & 0 & -\frac{a}{bc} \\ 0 & 1 & -\frac{a}{bc} & 0 \\ 0 & \frac{a}{bc} & 1 & 0 \\ \frac{a}{bc} & 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

[1]Note that in [58] the form of the $\mathcal{SU}(1,1)$ matrix of Equation 3.5 has already had the dimensional scales collected and so is in the form of Equation 3.11.

is a group element of $\mathcal{U}(1)$. The parameter v has units of velocity, f has units of force and r has units of power. The factor γ_{vfr} which appears in Equation 3.5 is the reciprocal counterpart to the gamma factor or "Lorentz factor" of special relativity. It is defined as:

$$\gamma_{vfr} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2 - \left(\frac{f}{b}\right)^2 + \left(\frac{r}{bc}\right)^2}}. \tag{3.7}$$

There are various limiting forms of this factor defined, for example, by:

$$\gamma_{vf} = \lim_{r \to 0} \gamma_{vfr},$$

and so on. In this way, the usual gamma factor of special relativity is denoted by γ_v and it is defined as:

$$\gamma_v = \lim_{f,r \to 0} \gamma_{vfr} = \lim_{b,c \to \infty} \gamma_{vfr}. \tag{3.8}$$

The full gamma factor of the quaplectic group is defined as:

$$\gamma_{vfra} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2 - \left(\frac{f}{b}\right)^2 + \left(\frac{r}{bc}\right)^2}\sqrt{1 + \left(\frac{a}{bc}\right)^2}}. \tag{3.9}$$

The matrix in Equation 3.5 will induce the following transformations of 1-phase space frame:

$$\begin{pmatrix} \frac{t'}{\lambda_t} \\ \frac{x'}{\lambda_x} \\ \frac{p'}{\lambda_p} \\ \frac{e'}{\lambda_e} \end{pmatrix} = \gamma_{vfr} \begin{pmatrix} 1 & \frac{v}{c} & \frac{f}{b} & -\frac{r}{bc} \\ \frac{v}{c} & 1 & \frac{r}{bc} & -\frac{f}{b} \\ \frac{f}{b} & -\frac{r}{bc} & 1 & \frac{v}{c} \\ \frac{r}{bc} & -\frac{f}{b} & \frac{v}{c} & 1 \end{pmatrix} \begin{pmatrix} \frac{t}{\lambda_t} \\ \frac{x}{\lambda_x} \\ \frac{p}{\lambda_p} \\ \frac{e}{\lambda_e} \end{pmatrix}, \tag{3.10}$$

where the dashes indicate the transformed frame. The dimensional scales λ_i can be multiplied through the rows and divided through the columns, leaving:

$$\begin{pmatrix} t' \\ x' \\ p' \\ e' \end{pmatrix} = \gamma_{vfr} \begin{pmatrix} 1 & \frac{v}{c^2} & \frac{f}{b^2} & -\frac{r}{b^2c^2} \\ v & 1 & \frac{r}{b^2} & -\frac{f}{b^2} \\ f & -\frac{r}{c^2} & 1 & \frac{v}{c^2} \\ r & -f & v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ p \\ e \end{pmatrix}. \tag{3.11}$$

Similarly, $\Xi(a)$ can be modified in the same way and becomes:

$$\Xi^\circ(a) = \frac{1}{\sqrt{1 + \left(\frac{a}{bc}\right)^2}} \begin{pmatrix} 1 & 0 & 0 & -\frac{a}{b^2c^2} \\ 0 & 1 & -\frac{a}{b^2} & 0 \\ 0 & \frac{a}{c^2} & 1 & 0 \\ a & 0 & 0 & 1 \end{pmatrix}. \tag{3.12}$$

3.1.3 Quaplectic Algebra

The matrices in Equations 3.11 and 3.12 can be differentiated with respect to each group parameter in order to define the generators of the quaplectic Lie algebra. The generators are defined as:

$$K := i\left.\frac{\partial \Xi}{\partial v}\right|_{v=0}, \qquad N := i\left.\frac{\partial \Xi}{\partial f}\right|_{f=0}$$
$$R := i\left.\frac{\partial \Xi}{\partial r}\right|_{r=0}, \qquad A := i\left.\frac{\partial \Xi^{\circ}}{\partial a}\right|_{a=0} = \left.\frac{\partial \Xi(v,f,r,a)}{\partial a}\right|_{a=0}.$$

As in Section 2.6, the generators are artificially multiplied by i in order to obtain Hermitian representations of the generators. A general element of the homogeneous algebra can therefore be written as:

$$d\Xi(v,f,r,a) = dvK + dfN + drR + daA. \tag{3.13}$$

As 4×4 matrices, the generators are:

$$K = \begin{pmatrix} 0 & \frac{i}{c^2} & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{c^2} \\ 0 & 0 & i & 0 \end{pmatrix} \qquad N = \begin{pmatrix} 0 & 0 & \frac{i}{b^2} & 0 \\ 0 & 0 & 0 & -\frac{i}{b^2} \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{3.14a}$$

$$R = \begin{pmatrix} 0 & 0 & 0 & -\frac{i}{b^2c^2} \\ 0 & 0 & \frac{i}{b^2} & 0 \\ 0 & -\frac{i}{c^2} & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 0 & 0 & -\frac{i}{b^2c^2} \\ 0 & 0 & -\frac{i}{b^2} & 0 \\ 0 & \frac{i}{c^2} & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}. \tag{3.14b}$$

These generators obey the following commutation relations:

$$[K,N] = 2iR \qquad [K,R] = \tfrac{2i}{c^2}N \qquad [K,A] = 0 \tag{3.15a}$$
$$[N,R] = -\tfrac{2i}{b^2}K \qquad [N,A] = 0 \qquad [R,A] = 0. \tag{3.15b}$$

These generators - with a suitable embedding of the 4×4 matrices into 6×6 matrices - can then be combined with the Heisenberg algebra generators realised in Equations 2.66 with the commutation relation given in Equation 2.63 to form the Lie algebra of the quaplectic group. The constants b and c can be inserted into Equation 3.13 in order to make the dimensions explicit. For example, dv has dimensions of ms^{-1} and so can be divided by c, whereas K has dimensions of $m^{-1}s$ and so can be multiplied by c. With this in mind, and combining Ξ with a general element dZ of the algebra of the Heisenberg group (cf. Equation 2.62), a general element of the quaplectic algebra is:

$$\begin{aligned} dQ =& d\Xi + dZ \\ =& \frac{dv}{c}cK + \frac{df}{b}bN + \frac{dr}{bc}bcR + \frac{da}{bc}bcA \\ &+ \frac{1}{\hbar}(deT + dpX + dxP + dtE) + \theta I. \end{aligned} \tag{3.16}$$

The commutation relations, along with the canonical commutation relations of Equations 2.63 and 3.15, are:

$$[K,T] = \tfrac{i}{c^2}X \qquad [K,X] = iT \qquad [K,P] = \tfrac{i}{c^2}E \qquad [K,E] = iP \tag{3.17a}$$
$$[N,T] = -\tfrac{i}{b^2}P \qquad [N,X] = \tfrac{i}{b^2}E \qquad [N,P] = -iT \qquad [N,E] = iX \tag{3.17b}$$
$$[R,T] = -\tfrac{i}{b^2c^2}E \qquad [R,X] = \tfrac{i}{b^2}P \qquad [R,P] = -\tfrac{i}{c^2}X \qquad [R,E] = iT \tag{3.17c}$$
$$[A,T] = -\tfrac{i}{b^2c^2}E \qquad [A,X] = -\tfrac{i}{b^2}P \qquad [A,P] = \tfrac{i}{c^2}X \qquad [A,E] = iT. \tag{3.17d}$$

3.1.3.1 n-Phase Space

In n-phase space the unitary irreducible representations of the Heisenberg group are defined in Equation 2.54 with the generators of the algebra defined in Equations 2.55 to 2.57. Following [43], a different set of Heisenberg generators can be obtained by taking the following combinations of the $\{X^\mu, P^\mu\}$ generators:

$$Z^\mu := \frac{1}{\sqrt{2}}\left(X^\mu - iP^\mu\right), \qquad \overline{Z}^\mu := \frac{1}{\sqrt{2}}\left(X^\mu + iP^\mu\right). \tag{3.18}$$

The position and momentum operators can be expressed in terms of these generators as:

$$X^\mu = \frac{1}{\sqrt{2}}(Z^\mu + \overline{Z}^\mu), \qquad P^\mu = \frac{i}{\sqrt{2}}(Z^\mu - \overline{Z}^\mu). \tag{3.19}$$

Together with the I operator of Equation 2.56, the Z^μ and $\overline{Z}^\mu$ generators obey the following commutation relations:

$$\left[Z^\mu, \overline{Z}_\nu\right] = -\eta^\mu{}_\nu I, \tag{3.20}$$

with all other combinations vanishing. The unitary representations of the generators of the homogeneous $\mathcal{U}(1,n)$ algebra $E^\mu{}_\nu$ obey the condition:

$$(E^\mu{}_\nu)^\dagger = \eta^{\mu\rho}\eta_{\nu\sigma}E^\sigma{}_\rho,$$

and the commutation relations:

$$[E^\mu{}_\nu, E^\rho{}_\sigma] = (\delta_\nu{}^\rho E^\mu{}_\sigma - \delta^\mu{}_\sigma E^\rho{}_\nu) \tag{3.21a}$$
$$[E^\mu{}_\nu, Z^\rho] = -\eta^{\mu\rho}Z_\nu \tag{3.21b}$$
$$\left[E^\mu{}_\nu, \overline{Z}^\rho\right] = \delta_\nu{}^\rho\overline{Z}^\mu, \tag{3.21c}$$

and the generator

$$U := \eta^{\mu\nu}E_{\mu\nu} = E^\mu{}_\mu \tag{3.22}$$

is the generator of the $\mathcal{U}(1)$ group [58]. Still following [43], auxiliary generators $\varepsilon^\mu{}_\nu$ can be defined from the $E^\mu{}_\nu$ generators using the anticommutator of Z and $\overline{Z}$ generators to make explicit a "spin-orbit" type decomposition. Define:

$$\rho(E^\mu{}_\nu) := Z^\mu{}_\nu = \tfrac{1}{2}\left\{\overline{Z}^\mu, Z_\nu\right\} \tag{3.23}$$

and auxiliary generators

$$W^{\mu}{}_{\nu} \equiv E^{\mu}{}_{\nu} - Z^{\mu}{}_{\nu}, \tag{3.24}$$

where ρ is a projective representation of elements of elements of $\mathcal{Q}(1,n)$ such that:

$$\rho(g) : \mathtt{H}^{\xi} \to \mathtt{H}^{\xi} \qquad \forall g \in \mathcal{G}(1,n)$$

(see Section 2.3.1, and note that there is only one little group for the quaplectic group and hence only one stabiliser group $\mathcal{G}^{\xi}$ which is the whole quaplectic group $\mathcal{Q}(1,n)$). A simple rearrangement of Equation 3.24 leads to:

$$E^{\mu}{}_{\nu} \equiv W^{\mu}{}_{\nu} + Z^{\mu}{}_{\nu}. \tag{3.25}$$

According to [54] and following [41], the process of finding the unitary irreducible representations of the quaplectic group as a semidirect product group (the process described in Section 2.3) reduces to this definition of a tensor product space of representations $Z^{\mu}{}_{\nu}$ of the quaplectic group and a "spin" of $\mathcal{U}(1,n)$ type, $W^{\mu}{}_{\nu}$. In terms of representations, Equation 3.25 can be rewritten as (cf. Sections 2.3 and 2.4 for the definitions of the representations τ, ρ and ξ):

$$\tau'(E^{\mu}{}_{\nu}) = \sigma'(W^{\mu}{}_{\nu}) + \rho'(Z^{\mu}{}_{\nu}), \tag{3.26}$$

where the apostrophes indicate the derivative of a group representation with respect to the group parameters at the identity of that parameter, which is a representation of an element of the algebra of the group. In order to simplify the notation, the representations of the generators in 3.26 will be replaced by the notation of the abstract generators themselves, with the exception of $\sigma'(W^{\mu}{}_{\nu})$ which will be represented by $\varepsilon^{\mu}{}_{\nu}$:

$$E^{\mu}{}_{\nu} \equiv \varepsilon^{\mu}{}_{\nu} + Z^{\mu}{}_{\nu}. \tag{3.27}$$

These generators satisfy the following commutation relations:

$$[E^{\mu}{}_{\nu}, \varepsilon^{\rho}{}_{\sigma}] = \delta^{\mu}{}_{\sigma}\varepsilon^{\rho}{}_{\nu} - \delta_{\nu}{}^{\rho}\varepsilon^{\mu}{}_{\sigma} \tag{3.28a}$$

$$[\varepsilon^{\mu}{}_{\nu}, \varepsilon^{\rho}{}_{\sigma}] = \delta^{\mu}{}_{\sigma}\varepsilon^{\rho}{}_{\nu} - \delta_{\nu}{}^{\rho}\varepsilon^{\mu}{}_{\sigma} \tag{3.28b}$$

$$[\varepsilon^{\mu}{}_{\nu}, Z^{\rho}] = \left[\varepsilon^{\mu}{}_{\nu}, \overline{Z}^{\rho}\right] = 0. \tag{3.28c}$$

Therefore the auxiliary generators $\varepsilon^{\mu}{}_{\nu}$ themselves satisfy the $\mathcal{U}(1,n)$ algebra but commute with the Heisenberg algebra $\mathcal{H}(n{+}1)$. This means that Casimir operators C_n of Gel'Fand type of the auxiliary generators are also Casimir operators of the quaplectic group. These Casimir operators are defined in Section 3.1.4.

Poincaré generators $L_{\mu\nu}$ and "reciprocal boost" generators $M_{\mu\nu}$ can be defined in terms of the quaplectic generators $E^{\mu}{}_{\nu}$ as follows:

$$L_{\mu\nu} = i(E_{\mu\nu} - E_{\nu\mu}), \qquad M_{\mu\nu} = E_{\mu\nu} + E_{\nu\mu}.$$

These generators, along with the generators X_μ and P_μ of the Heisenberg algebra, obey the following commutation relations:

$$[L_{\rho\sigma}, L_{\mu\nu}] = i(\eta_{\sigma\mu}L_{\rho\nu} - \eta_{\rho\mu}L_{\sigma\nu} - \eta_{\sigma\nu}L_{\rho\mu} + \eta_{\rho\nu}L_{\sigma\mu}) \tag{3.29a}$$
$$[L_{\rho\sigma}, M_{\mu\nu}] = i(\eta_{\sigma\mu}M_{\rho\nu} - \eta_{\rho\mu}M_{\sigma\nu} + \eta_{\sigma\nu}M_{\rho\mu} - \eta_{\rho\nu}M_{\sigma\mu}) \tag{3.29b}$$
$$[M_{\rho\sigma}, M_{\mu\nu}] = i(\eta_{\sigma\mu}L_{\rho\nu} + \eta_{\rho\mu}L_{\sigma\nu} + \eta_{\sigma\nu}L_{\rho\mu} + \eta_{\rho\nu}L_{\sigma\mu}) \tag{3.29c}$$
$$[L_{\rho\sigma}, X_\mu] = i(\eta_{\sigma\mu}X_\rho - \eta_{\rho\mu}X_\sigma) \tag{3.29d}$$
$$[L_{\rho\sigma}, P_\mu] = i(\eta_{\sigma\mu}P_\rho - \eta_{\rho\mu}P_\sigma) \tag{3.29e}$$
$$[M_{\rho\sigma}, X_\mu] = -i(\eta_{\sigma\mu}P_\rho + \eta_{\rho\mu}P_\sigma) \tag{3.29f}$$
$$[M_{\rho\sigma}, P_\mu] = i(\eta_{\sigma\mu}X_\rho + \eta_{\rho\mu}X_\sigma). \tag{3.29g}$$

Note that the set of generators $L_{\mu\nu}$ together with generators P_μ form a Poincaré subalgebra of the quaplectic group. Additionally, the generators $L_{\mu\nu}$ together with generators X_μ also form a Poincaré subalgebra. In fact, generators Y_μ could be defined with time as the timelike component and momentum as the spacelike components, or generators Q_μ with energy as the timelike component and position as the spacelike components. These sets of generators together with the $L_{\mu\nu}$ generators would each form a Poincaré subalgebra. In this way it can be seen that the quaplectic group contains four Poincaré subgroups.

The $L_{\mu\nu}$ and $M_{\mu\nu}$ generators can be further separated into generators which mix spacelike with timelike components and those that purely involve spacelike components as follows:

$$J_{ij} := L_{ij} = -L_{ji} \qquad K_i := \frac{L_{0i}}{c} = -\frac{L_{i0}}{c} \tag{3.30a}$$
$$R_{ij} := \frac{M_{ij}}{bc} = \frac{M_{ji}}{bc} \qquad N_i := \frac{M_{0i}}{b} = \frac{M_{i0}}{b} \tag{3.30b}$$
$$A := \frac{M_{00}}{bc}. \tag{3.30c}$$

The Heisenberg generators can also be separated into timelike and spacelike components as follows:

$$X_\mu := (\tfrac{T}{\lambda_t}, \tfrac{X_i}{\lambda_x})^T \qquad P_\mu := (\tfrac{E}{\lambda_e}, \tfrac{-P_i}{\lambda_p})^T. \tag{3.31}$$

so that:

$$T := X_0\lambda_t \qquad X_i := X_i\lambda_x \qquad E := P_0\lambda_e \qquad P_i := -P_i\lambda_p.$$

A general element of the algebra (cf. Equation 3.16) $d\mathfrak{Q}$ can therefore be written as:

$$\begin{aligned} d\mathfrak{Q} =& d\alpha^{ij}J_{ij} + dv^iK_i + df^iF_i + dr^{ij}R_{ij} + daA \\ &+ \tfrac{1}{\hbar}(deT + dpX - dxP + dtE) + \tfrac{1}{i}d\theta I \end{aligned} \tag{3.32a}$$
$$\begin{aligned} d\mathfrak{Q} =& d\alpha^{ij}J_{ij} + \tfrac{dv^i}{c}cK_i + \tfrac{df^i}{b}bF_i + \tfrac{dr^{ij}}{bc}bcR_{ij} + \tfrac{da}{bc}bcA \\ &+ (\tfrac{de}{\lambda_e}\tfrac{T}{\lambda_t} + \tfrac{dp}{\lambda_p}\tfrac{X}{\lambda_x} - \tfrac{dx}{\lambda_x}\tfrac{P}{\lambda_p} + \tfrac{dt}{\lambda_t}\tfrac{E}{\lambda_e}) + \tfrac{1}{i}d\theta I. \end{aligned} \tag{3.32b}$$

The constants b and c and the dimensional scales have been made explicit in Equation 3.32b in order to show the dimensional relationships between the various generators and hence to be able to write the commutation relations with constants b and c included. The algebra of these generators can then be derived from Equations 3.29. The commutation relations of the homogeneous generators are:

$$\begin{aligned}
[J_{ij}, J_{kl}] &= i(\delta_{jk}J_{il} - \delta_{jk}J_{jl} - \delta_{jl}J_{ik} + \delta_{il}J_{jk}) \\
[J_{ij}, K_k] &= i(\delta_{jk}K_i - \delta_{ik}K_j) \\
[K_i, K_j] &= \tfrac{i}{c^2}J_{ij} \\
[J_{ij}, R_{kl}] &= i(\delta_{jk}R_{il} - \delta_{jk}R_{jl} + \delta_{jl}R_{ik} - \delta_{il}R_{jk}) \\
[J_{ij}, N_k] &= i(R_{jk} + \delta_{jk}A) \\
[J_{ij}, A] &= 0 \\
[K_i, R_{jk}] &= \tfrac{i}{c^2}(\delta_{ij}N_k + \delta_{ik}N_j) \\
[K_i, N_j] &= i(R_{ij} + \delta_{ij}A) \\
[K_i, A] &= \tfrac{2i}{c^2}N_i \\
[R_{ij}, R_{kl}] &= \tfrac{i}{b^2c^2}(\delta_{jk}J_{il} + \delta_{jk}J_{jl} + \delta_{jl}J_{ik} + \delta_{il}J_{jk}) \\
[R_{ij}, N_k] &= \tfrac{i}{b^2}(-\delta_{jk}K_i + \delta_{ik}K_j) \\
[R_{ij}, A] &= 0 \\
[N_i, N_j] &= -\tfrac{i}{b^2}J_{ij} \\
[N_i, A] &= \tfrac{2i}{b^2}K_i.
\end{aligned}$$

The commutation relations of the Heisenberg generators are given in Equation 2.63. The remaining commutation relations are:

$$\begin{aligned}
[J_{ij}, T] &= 0 & [J_{ij}, X_k] &= i(\delta_{jk}X_i - \delta_{ik}X_j) \\
[J_{ij}, E] &= 0 & [J_{ij}, P_k] &= i(\delta_{jk}P_i - \delta_{ik}P_j) \\
[K_i, T] &= \tfrac{i}{c^2}X_i & [K_i, X_j] &= i\delta_{ij}T \\
[K_i, E] &= -iP_i & [K_i, P_j] &= -\tfrac{i}{c^2}E \\
[R_{ij}, T] &= 0 & [R_{ij}, X_k] &= \tfrac{i}{b^2}(\delta_{jk}P_i - \delta_{ik}P_j) \\
[R_{ij}, E] &= 0 & [R_{ij}, P_k] &= -\tfrac{i}{c^2}(\delta_{jk}X_i - \delta_{ij}X_j) \\
[N_i, T] &= -\tfrac{i}{b^2}P_i & [N_i, X_j] &= -\tfrac{i}{b^2}\delta_{ij}E \\
[N_i, E] &= -iX_i & [N_i, P_j] &= -i\delta_{ij}T \\
[A, T] &= 0 & [A, X_i] &= 0 \\
[A, E] &= 0 & [A, P_i] &= 0.
\end{aligned}$$

If there is only one spatial dimension then there will be no J_{ij} generators and the generators K_1 and N_1 are identified with the K and N of Equation 3.14a respectively.

3.1.4 Quaplectic Casimir Operators

The auxiliary generators defined in Equation 3.24 allow an easy method to determine the Casimir invariant operators of the quaplectic group. With the following

definitions:

$$(\varepsilon^{(n+1)})^{\mu}{}_{\nu} := (\varepsilon^{(n)})^{\mu}{}_{\rho}\varepsilon^{\rho}{}_{\nu}, \qquad (\varepsilon^{(1)})^{\mu}{}_{\nu} := \varepsilon^{\mu}{}_{\nu}, \tag{3.33}$$

The Casimir operators C_n can be identified as:

$$C_n = \mathrm{tr}(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\mu}{}_{\mu}.$$

Recalling from Equation 3.22 that $U = E^{\mu}{}_{\mu}$ is the $\mathcal{U}(1)$ generator, and noting that the central generator I of the Heisenberg group is a trivial Casimir operator of the quaplectic group:

$$C_0 = I;$$

the second Casimir C_1 is given by:

$$\begin{aligned} C_1 &= E^{\mu}{}_{\mu} - Z^{\mu}{}_{\mu} \\ &= UI - \tfrac{1}{2}(P^{\mu}P_{\mu} + X^{\mu}X_{\mu}). \end{aligned}$$

There are five independent Casimir operators [58]. Just as for the Poincaré group and the Klein-Gordon equation 2.8.4.3, the eigenvalue equations of the unitary irreducible representations of these Casimir operators give rise to field equations; field equations which should hopefully correspond to physical phenomena. This problem has been attempted by Low in [56] and elsewhere, but is not repeated here.

3.2 Reciprocal Relativity

The Lorentz transformations in $(1+1)$ dimensional spacetime are:

$$\begin{aligned} t' &= \gamma_v \left(t + \frac{v}{c^2}x\right) \\ x' &= \gamma_v \left(x + vt\right), \end{aligned}$$

where

$$\gamma_v = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

is the usual gamma factor of special relativity (defined in Equation 3.8 as a limiting form of γ_{vfr}). The reciprocally relativistic analogue to these transformations are given from Equation 3.11 as:

$$dt' = \gamma_{vfr}\left(dt + \tfrac{v}{c^2}dx + \tfrac{f}{b^2}dp - \tfrac{r}{b^2c^2}de\right) \tag{3.34}$$

$$dx' = \gamma_{vfr}\left(dx + vdt + \tfrac{r}{b^2}dp - \tfrac{f}{b^2}de\right) \tag{3.35}$$

$$dp' = \gamma_{vfr}\left(dp + fdt - \tfrac{r}{c^2}dx + \tfrac{v}{c^2}de\right) \tag{3.36}$$

$$de' = \gamma_{vfr}\left(de + vdp - fdx + rdt\right), \tag{3.37}$$

where γ_{vfr} is defined in Equation 3.7. The multiplication of two Ξ matrices of the form of Equation 3.11 defines the velocity, force and addition laws. For $\Xi(v'', f'', r'') = \Xi(v', f', r') \cdot \Xi(v, f, r)$:

$$v'' = \frac{v + v' + \frac{1}{b^2}(f'r - fr')}{1 + \frac{vv'}{c^2} + \frac{ff'}{b^2} - \frac{rr'}{b^2c^2}} \tag{3.38}$$

$$f'' = \frac{f + f' + \frac{1}{c^2}(vr' - v'r)}{1 + \frac{vv'}{c^2} + \frac{ff'}{b^2} - \frac{rr'}{b^2c^2}} \tag{3.39}$$

$$r'' = \frac{r + r' + vf' - v'f}{1 + \frac{vv'}{c^2} + \frac{ff'}{b^2} - \frac{rr'}{b^2c^2}} \tag{3.40}$$

The notions of time dilation and length contraction can be modified to reflect these transformations. Consider two observers, $\mathcal{O}$ and $\mathcal{O}'$. Note that neither of the observers' reference frames are said to be at rest, and neither of the observers is said to be accelerating whilst the other is not. On the contrary, the observers have a relative velocity v, as well as a relative force (or acceleration) f and a relative power r. In other words, there are no restrictions placed upon how the two reference frames relate to one another.

Consider a clock at rest in the reference frame of observer $\mathcal{O}$. This observer measures an infinitesimal period of time with this clock and labels it $dt := dt_2 - dt_1$. The observer $\mathcal{O}'$ will measure a corresponding time and label it $dt' := dt'_2 - dt'_1$. From equation 3.34:

$$\begin{aligned} dt' &= dt'_2 - dt'_1 \\ &= \gamma_{vfr}\left(dt_2 - dt_1 + \frac{v}{c^2}(dx_2 - dx_1) + \frac{f}{b^2}(dp_2 - dp_1) - \frac{r}{b^2c^2}(de_2 - de_1)\right) \end{aligned}$$

However, since the clock is at rest in $\mathcal{O}$'s reference frame, $dx_2 = dx_1$, $dp_2 = dp_1$ and $de_2 = de_1$. Therefore, the time measured by $\mathcal{O}'$ is given by:

$$dt' = \gamma_{vfr} dt. \tag{3.41}$$

This is the reciprocally relativistic analogue of time dilation. Now consider a rod at rest in the reference frame of observer $\mathcal{O}$. This observer measures the clock to have a length of $L := x_2 - x_1$. The observer $\mathcal{O}'$ will measure a corresponding length of $L' := x'_2 - x'_1$. From equation 3.35:

$$x_2 - x_1 = \gamma_{vfr}\left(x'_2 - x'_1 - v(t'_2 - t'_1) - \frac{r}{b^2}(p'_2 - p'_1) + \frac{f}{b^2}(e'_2 - e'_1)\right)$$

If $\mathcal{O}'$ is to make the measurement accurately, he needs to measure x'_2 and x'_1 simultaneously. Therefore, $t'_2 = t'_1$, $p'_2 = p'_1$ and $e'_2 = e'_1$, and $\mathcal{O}'$ will measure the rod to have length:

$$L' = \frac{L}{\gamma_{vfr}(v(t), f(t), r(t))}. \tag{3.42}$$

Thus the classical results of time dilation and length contraction are also features of reciprocal relativity with a few differences: in reciprocal relativity, the usual γ_v factor as given above is replaced by γ_{vfr}, which is a function of v, f and r - variables which may all vary with time. In addition, the frames in question no longer need to be inertial; the result is true for *all* reference frames. But perhaps the most important difference is the fact that γ_{vfr} is itself a function of time since the velocity changes as one frame accelerates relative to the other. This means that $\mathcal{O}'$ will measure the rod to have different lengths at different times, but more interestingly it means that Equation 3.41 is only true for infinitesimal periods of time. To properly measure the time dilation therefore requires Equation 3.41 to be integrated over the period of time in question:

$$t' = \int \gamma_{vfr}(v(t), f(t), r(t))dt. \tag{3.43}$$

The simplest example is of an observer $\mathcal{O}$ moving with a constant acceleration relative to $\mathcal{O}'$ with constant mass and a zero initial relative velocity. Then:

$$\begin{aligned} v &= at \\ f &= ma \\ r &= \tfrac{de}{dt} = f\tfrac{dx}{dt} = fv = fat, \end{aligned}$$

and so Equation 3.43 becomes:

$$\begin{aligned} t' &= \int \frac{1}{\sqrt{1-(\frac{v(t)}{c})^2-(\frac{f}{b})^2+(\frac{fv(t)}{bc})^2}}dt \\ &= \int \frac{1}{\sqrt{1-(\frac{at}{c})^2-(\frac{f}{b})^2+(\frac{fat}{bc})^2}}dt \\ &= \int \frac{1}{\sqrt{(1-(\frac{f}{b})^2)(1-(\frac{a}{c})^2t^2)}}dt \\ &= \frac{1}{\sqrt{1-(\frac{f}{b})^2}}\frac{c}{t}\arctan\left(\frac{\frac{at}{c}}{\sqrt{1-(\frac{at}{c})^2}}\right) \\ &= \gamma_f\frac{c}{t}\arctan\left(\frac{\frac{at}{c}}{\sqrt{1-(\frac{at}{c})^2}}\right), \end{aligned}$$

where γ_f is constant in time.

3.2.1 Proper Acceleration

In [58] Low has worked out the reciprocal relativistic analogue of proper acceleration or *Rindler acceleration*. Take three frames: S, S'' and S°. At a particular

point in time, S has the same velocity, force and power relative to S° as does S''. Therefore, from Equations 3.38, 3.39 and 3.40:

$$\begin{aligned} v'' &= v \\ f'' &= f \\ r'' &= r. \end{aligned}$$

In other words, $v' = f' = r' = 0$, which is to say that there is no difference in the velocity, force and power of the frames S and S''. However, this is only true momentarily, and there may be a relative difference in the rate of change of the velocity, force and power of the frames S and S''. The frame S° is the "inertial rest frame" of S, and so the proper acceleration of S'' is measured relative to this frame S°. The derivative of Equation 3.38 is given by:

$$\begin{aligned} dv'' = dv + dv' + \tfrac{1}{b^2}(df'r + f'dr - dfr' - fdr')... \\ - [v + v' + \tfrac{1}{b^2}(f'r - fr')][\tfrac{1}{c^2}(dvv' + vdv') + \tfrac{1}{b^2}(df f' + fdf') - \tfrac{1}{b^2c^2}(drr' + rdr')] \end{aligned}$$

Imposing the following conditions $dv' = df' = dr' = 0$, $v = f = r = 0$, $v' = v''$, $f' = f''$, $r' = r''$ simplifies this derivative to:

$$\begin{aligned} dv'' &= dv + \tfrac{1}{b^2}(f'dr - dfr') - v'(\tfrac{dvv'}{c^2} + \tfrac{df f'}{b^2} - \tfrac{drr'}{b^2c^2}) \\ &= (1 - \tfrac{(v'')^2}{c^2})dv - \tfrac{1}{b^2}[(r'' + v''f'')df - (f'' + \tfrac{v''r''}{c^2})dr]. \end{aligned}$$

similarly for df'' and dr'':

$$\begin{aligned} df'' &= (1 - \tfrac{(f'')^2}{b^2})df + \tfrac{1}{c^2}[(r'' - v''f'')dv - (v'' - \tfrac{f''r''}{b^2})dr] \\ dr'' &= (1 + \tfrac{(r'')^2}{b^2c^2})dr + (f'' - \tfrac{v''r''}{c^2})dv - (v'' + \tfrac{f''r''}{b^2})df. \end{aligned}$$

These velocity, force and power transformations can be put into matrix form:

$$\begin{pmatrix} dv'' \\ df'' \\ dr'' \end{pmatrix} = \begin{pmatrix} 1 - \frac{(v'')^2}{c^2} & -(r'' + v''f'') & \frac{1}{b^2}(f'' + \frac{v''r''}{c^2}) \\ r'' - f''v'' & 1 - \frac{(f'')^2}{b^2} & \frac{1}{c^2}(-v'' + \frac{f''r''}{b^2}) \\ f'' - \frac{v''r''}{c^2} & -v'' + \frac{f''r''}{b^2} & 1 + \frac{(r'')^2}{b^2c^2} \end{pmatrix} \begin{pmatrix} dv' \\ df' \\ dr' \end{pmatrix},$$

and then inverted, to get:

$$\begin{pmatrix} dv' \\ df' \\ dr' \end{pmatrix} = \gamma_{vfr}^2(v''f''r'') \begin{pmatrix} 1 & \frac{r''}{b^2} & -\frac{f''}{b^2} \\ -\frac{r''}{c^2} & 1 & \frac{v''}{c^2} \\ -f'' & v'' & 1 \end{pmatrix} \begin{pmatrix} dv'' \\ df'' \\ dr'' \end{pmatrix}.$$

From Equation 3.41, using the present notation, $dt'' = \gamma_{vfr}dt'$. Therefore, the proper rates of change of velocity (or proper acceleration), force and power are given by:

$$\begin{aligned} \frac{dv'}{dt'} &= \gamma_{vfr}^3(v'', f'', r'')\left(\frac{dv''}{dt''} + \frac{r''}{b^2}\frac{df''}{dt''} - \frac{f''}{b^2}\frac{dr''}{dt''}\right) \\ \frac{df'}{dt'} &= \gamma_{vfr}^3(v'', f'', r'')\left(\frac{df''}{dt''} - \frac{r''}{c^2}\frac{dv''}{dt''} + \frac{v''}{c^2}\frac{dr''}{dt''}\right) \\ \frac{dr'}{dt'} &= \gamma_{vfr}^3(v'', f'', r'')\left(\frac{dr''}{dt''} - f''\frac{dv''}{dt''} + v''\frac{df''}{dt''}\right). \end{aligned}$$

Chapter 4

Group Contractions

Galilean mechanics is the inertial non-relativistic limit of special relativity. In other words, Galilean mechanics is actually a special case of a larger symmetry: that of special relativity. In group theoretical terms, this means that the Galilean group is an İnönü-Wigner contraction of the Poincaré group, which is the group of special relativity. In the same way, if Born reciprocity is true, then the current laws of physics which do not now appear to be reciprocally invariant must be a special case of a larger, reciprocally invariant, symmetry. The Galilean group as an İnönü-Wigner contraction of the Poincaré group is the main example studied in [40]. This contraction is carried out with respect to the group generated by rotations as well as displacements in time. In practice, the contraction is produced by taking the limit $c \to \infty$ in the representation of the group elements. The contraction method itself is discussed in Section 4.1, and the contraction of the Poincaré group to the Galilean group is described in Section 4.1.2. A simpler contraction is described first in Section 4.1.1; that of the Heisenberg group to the translation group. İnönü-Wigner contractions can also be carried out for the quaplectic group, leading to physically interesting subgroups. Taking the limits of $b, c \to \infty$ leads directly to the Hamilton group, and this is described in Section 4.1.3 (see Section 3.1 for the definition of b). Setting $f = r = 0$ in the Hamilton group - although not a group contraction - leads to the Galilean group, and this is described in Section 4.1.3.3 (f is the relative force between respective reference frames and r the relative power - see Section 3.1.2 for more detail). Taking the limit $b \to \infty$ for $f = r = 0$ in the quaplectic group leads to the Poincaré group, and this is described in Section 4.1.4. The contractions will be undertaken in the algebra of the homogeneous subgroup of the quaplectic group, since the Lie algebras of the homogeneous groups are easily represented by square matrices. However, in Section 4.1.2 the contraction has been undertaken in the algebra of the inhomogeneous Euclidean group (in order to follow the [40] paper). The semidirect product of the homogeneous group and an appropriate normal subgroup must then be taken to form the inhomogeneous group. The inhomogeneous group must then be centrally extended in order to arrive at the physically interesting group (cf. Sections 2.2.1 and 2.2.2). The potential contractions of the quaplectic group are summarised in the diagram in Equation 4.2.

In the following two diagrams, an arrow with an I indicates an extension of a

homogeneous group to an inhomogeneous group by taking the semidirect product with a normal subgroup, as per the process described in Section 2.3. An arrow with a C refers to taking the central extension of the group, as per the process described in Sections 2.2.1 and 2.2.2. Vertical arrows with a W on the left indicate İnönü-Wigner contractions. For the Heisenberg group:

$$\begin{array}{ccc} \mathcal{H}(n) & & \\ {\scriptstyle W}\big\downarrow{\scriptstyle \hbar\to 0} & & \\ \mathcal{T}(2n) & \xrightarrow{C} & \mathcal{H}(n) \\ {\scriptstyle I}\big\downarrow & & \\ \mathcal{H}(2n). & & \end{array} \tag{4.1}$$

Either a central extension or a semidirect product ($\mathcal{H}(n) \cong \mathcal{T}(n) \ltimes \mathcal{T}(n+1)$; cf. Section 2.6), will extend the translation group to the Heisenberg group. For the quaplectic group:

$$\begin{array}{ccccc} \mathcal{E}(n) & \xrightarrow{I} & \mathcal{E}(n) \ltimes \mathcal{T}(n+1) & \xrightarrow{C} & \mathcal{G}a(n) \\ \big\uparrow{\scriptstyle f=r=0} & & & & \\ \mathcal{H}a(n) & \xrightarrow{I} & \mathcal{H}a(n) \ltimes \mathcal{T}(n) & \xrightarrow{C} & \widehat{\mathcal{H}a}(n) \ltimes \mathcal{H}(n) \\ {\scriptstyle W}\big\uparrow{\scriptstyle b,c\to\infty} & & & & \\ \mathcal{U}(1,n) & \xrightarrow{I} & \mathcal{U}(1,n) \ltimes \mathcal{H}(n+1) & \xrightarrow{C} & \mathcal{Q}(1,n) \\ {\scriptstyle W}\big\downarrow{\scriptstyle b\to\infty,f=r=0} & & & & \\ \mathcal{L}(1,n) & \xrightarrow{I} & \mathcal{L}(1,n) \ltimes \mathcal{T}(n+1) & \xrightarrow{C} & \mathcal{P}(1,n) \\ {\scriptstyle W}\big\downarrow{\scriptstyle c\to\infty} & & & & \\ \mathcal{E}(n) & \xrightarrow{I} & \mathcal{E}(n) \ltimes \mathcal{T}(n+1) & \xrightarrow{C} & \mathcal{G}a(n). \end{array} \tag{4.2}$$

There are also other contractions which could be carried out, such as the limit $c \to \infty$ for $v = r = 0$. Recall that the contractions are carried out at the level of the homogeneous group, which is $\mathcal{U}(1,n)$ in the case of the quaplectic group. The contractions could also be carried out either at the level of the inhomogeneous groups or of the centrally extended groups - $\mathcal{Q}(1,n)$, $\mathcal{P}(1,n)$ and so on - however the latter are not easily represented by matrices and so the details of the contractions are not as simple.

4.1 Group Contractions

In their 1953 paper [40], İnönü and Wigner describe a general method to map the representations of one group into the representations of another group as a limiting case. For an arbitrary Lie group $\mathcal{G}$ with a general element $g(a^i)$, a^i, $i = 1, 2, ..., n$

and generators $X_i := \frac{\partial g}{\partial a^i}\big|_{a^i=0}$, the structure constants $C^k{}_{ij}$ are defined by (cf. Equation 2.8):

$$[X_i, X_j] = C^k{}_{ij} X^k. \tag{4.3}$$

The generators X_i can undergo a linear homogeneous transformation $U^i{}_j$ to define new generators Y_j:

$$Y_i := U_i{}^j X_j, \tag{4.4}$$

where

$$Y_i = \left.\frac{\partial g(b^i)}{\partial b^i}\right|_{b^i=0},$$

and so:

$$e^{b^i Y_i} = e^{b^i U_i{}^j X_j}$$

But $e^{b^i Y_i} = e^{a^i X_i}$, and so the group parameters a^i can be written as:

$$a^i = b^j U_j{}^i. \tag{4.5}$$

If this transformation is non-singular then the structure constants for the X^i generators will be replaced by other constants for the Y^i generators but the structure of the group will not change. If, however, the transformation is singular, then the transformation will lead to a new group. The new group is called the *contracted* group [40]. The contraction theorem, as stated by İnönü and Wigner in [40], is:

> Every Lie group can be contracted with respect to any of its continuous subgroups and only with respect to these. The subgroup with respect to which the contraction is undertaken will be called S. The contracted [generators] form an abelian invariant subgroup of the contracted group. The subgroup S with respect to which the contraction was undertaken is isomorphic with the factor group of this invariant subgroup. Conversely, the existence of an abelian invariant subgroup and the possibility to choose from each of its cosets an element so that these form a subgroup S, is a necessary condition for the possibility to obtain the group from another group by contraction.

The necessary and sufficient condition[1] that a group can be contracted is that the matrix U be able to be transformed by suitable nonsingular matrices α and β to the form:

$$\begin{aligned} U' &= \beta U \alpha^{-1} \\ &= u + \epsilon w, \end{aligned}$$

[1] This condition is defined in [40] but is much clearer in [72].

where

$$u := \begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}$$
$$w := \begin{pmatrix} v & 0 \\ 0 & I_{t\times t} \end{pmatrix}.$$

Henceforth, $U' = u + \epsilon w$ will be labelled simply as U. The matrix v in w is a square $r \times r$ matrix, so it has the same number of rows and columns as the identity matrix in u and $t = n - r$. The generators Y_i become, from Equation 4.4:

$$\begin{aligned} Y_i &= u^j{}_i X_j + \epsilon w^j{}_i X_j \\ &= X_j + \epsilon v^j{}_i X_j && \text{for } i = 1, ..., r \\ &= \epsilon X_i && \text{for } i = r+1, ..., n. \end{aligned}$$

Similarly, the group parameters a^i can be written, from Equation 4.5, as:

$$\begin{aligned} a^i &= u^i{}_j b^j + \epsilon w^i{}_j b^j \\ &= b^j + \epsilon v^i{}_j b^j && \text{for } i = 1, ..., r \\ &= \epsilon b^i && \text{for } i = r+1, ..., n. \end{aligned}$$

In [40], it is shown that the contracted structure constants $C^k{}_{ij}$ of Equation 4.3 must vanish for $k = r+1, ..., n$ in the limit $\epsilon \to 0$ in order for the commutators of the Y generators to converge. In other words, the generators X_i, $i = 1, ..., r$ span the subgroup, S, with respect to which the contraction is carried out. The generators X_i, $i = r+1, ..., n$ are the contracted generators. The subgroup S remains unchanged under the contraction and so the group parameters can be chosen such that they are equal to zero for $i = r+1, ..., n$ [40] throughout S. With this choice:

$$\begin{aligned} a^i &= b^i && \text{for } i = 1, ..., r \\ a^i &= \epsilon b^i && \text{for } i = r+1, ..., n. \end{aligned}$$

The matrix $U = u + \epsilon w$ becomes singular in the limit $\epsilon \to 0$, which is the limit that defines the contraction. Alternatively, one could define $\varepsilon = 1/\epsilon$, and the contraction is therefore defined in the limit $\varepsilon \to \infty$.

4.1.1 The Heisenberg Group Contracts to the Translation Group

The Heisenberg group contracts to the translation group as $\hbar \to 0$. The contraction is presented here first in order to show how the method works in a simple case.

The Heisenberg group in $(1+1)$ dimensions can be realised by matrices of the form of Equation 2.60. The limit $\hbar \to 0$ can then be taken in order to effect the

İnönü-Wigner contraction. The resulting matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -e & p & -x & t & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

This matrix is an abelian matrix, and a representation of the translation group $\mathcal{T}(4)$ involving translations in space, time, energy and momentum. The last row and column can be omitted, and the matrix can instead be written, as its symplectic transpose (see Section 1.3), as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & t \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & p \\ 0 & 0 & 0 & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The corresponding generators of the Lie algebra can be determined by taking the derivative of this matrix with respect to each parameter at the identity. The generators are:

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.6a}$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.6b}$$

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.6c}$$

$$T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.6d}$$

Clearly, all the matrices commute. This is to be expected since the group is abelian. This İnönü-Wigner contraction is the reversal of the process of taking the central extension of the translation group.

4.1.2 The Poincaré Group Contracts to the Galilean Group

The first example of a group contraction studied in [40] is that of the Poincaré group to the Galilean group in $(1+1)$ dimensions. This contraction in [40] is reproduced here, firstly to show how the contraction method works in a slightly more difficult context, but also to demonstrate how the mechanics of theoretical group theory is representative of real physics: the limit $c \to \infty$ takes one from the realm of special relativity back to the realm of inertial Newtonian mechanics. So too the Poincaré group (really the inhomogeneous Lorentz group) is contracted to the Galilean group (the inhomogeneous Euclidean group) in the limit. This is to be used as an analogy for the quaplectic group. The contraction is carried out at the level of the inhomogeneous group in order to reproduce the method of [40]. The same contraction could equally well be carried out at the level of the homogeneous group. The semi-direct product of the resulting contracted group - the homogeneous Euclidean group - with the translation group would yield the inhomogeneous Euclidean group; so either route can be taken. The Galilean group is then given by the central extension of the inhomogeneous Euclidean group. It is the unitary representations of the Galilean group which correspond to physical states, which are rays in a Hilbert space (cf. Sections 2.2, 2.2.1 and 2.2.2).

In $(1+1)$ dimensional spacetime the inhomogeneous Lorentz transformations can be represented by the (non-unitary) matrices:

$$\Gamma(v, a_i) = \begin{pmatrix} \Lambda(v) & a_i \\ 0 & 1 \end{pmatrix} \tag{4.7a}$$

$$= \begin{pmatrix} 1 & \frac{v}{c} & \frac{a_0}{\lambda_t} \\ \frac{v}{c} & 1 & \frac{a_1}{\lambda_x} \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.7b}$$

where v is the relative velocity between two reference frames, a_0 is a translation in time and is divided by λ_t so that each of the terms involved in the representations remain dimensionless while a_0 itself is measured in units of λ_t. Similarly, a_1 is a translation in space and is measured in units of λ_x; therefore it is similarly divided by λ_x in order for the term to be dimensionless. These two translations are then denoted by the tuplet a_i, $i = 0, 1$ in Equation 4.7a. $\Lambda(v)$ is a Lorentz transformation. The $\Gamma(v, a_i)$ matrix defines transformations on the spacetime frame (t, x):

$$\begin{aligned} \tfrac{t'}{\lambda_t} &= \tfrac{t}{\lambda_t} + \tfrac{vx}{c\lambda_x} + \tfrac{a_0}{\lambda_t} \\ \Rightarrow t' &= t + \tfrac{v}{c^2}x + a_0 \end{aligned}$$

and

$$\begin{aligned} \tfrac{x'}{\lambda_x} &= \tfrac{vt}{c\lambda_t} + \tfrac{x}{\lambda_x} + \tfrac{a_1}{\lambda_x} \\ \Rightarrow x' &= vt + x + a_1. \end{aligned}$$

The second lines for each transformation could be defined direct from the matrix in Equation 4.7b by multiplying the first row and dividing the first column by

λ_t and multiplying the second row and dividing the second column by λ_x. This defines a dimensional matrix $\Gamma(v, a_i)$ (by abuse of notation it will be denoted in the same way as the dimensionless matrix) as:

$$\Gamma(v, a_i) = \begin{pmatrix} 1 & \frac{v}{c^2} & a_0 \\ v & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.8}$$

The generators of the Lie algebra for this group can be defined by taking the derivative of Γ with respect to each parameter v, a_0 and a_1 at the identity and multiplying by i (cf. Section 2.1.3.1). These generators are:

$$K := i \left.\frac{\partial \Gamma(v, a_i)}{\partial v}\right|_{v=0} = \begin{pmatrix} 0 & \frac{i}{c^2} & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H := i \left.\frac{\partial \Gamma(v, a_i)}{\partial a_0}\right|_{a_0=0} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$P := i \left.\frac{\partial \Gamma(v, a_i)}{\partial a_1}\right|_{a_1=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & 0 & 0 \end{pmatrix}$$

The factor of i is introduced in order to make the generators Hermitian rather than skew-Hermitian (cf. Section 2.1.3). The K generator is a boost in spacetime; the H generator is a translation in time or energy operator, and the P generator is a translation in space or momentum operator. The commutation relations for these generators are:

$$[K, H] = iP \qquad [K, P] = \frac{i}{c^2} H \qquad [H, P] = 0. \tag{4.9}$$

The contraction is then carried out by taking the limit $c \to \infty$. Under this limit the transformation becomes:

$$\begin{pmatrix} t' \\ x' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_0 \\ v & 1 & a_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ 1 \end{pmatrix}. \tag{4.10}$$

In other words, the time and space coordinates transform as:

$$t \to t' = t + a_0$$
$$x \to x' = x + vt + a_1.$$

These are the transformations of the inhomogeneous Euclidean group in 1 dimension, $\mathcal{E}(1) \ltimes \mathcal{T}(2)$. As for the Lie algebra, the generators P and H remain the same in the limit as $c \to \infty$; however the boost generator K does not. Define the

Galilean boost generator G to be the limit of K as $c \to \infty$. A matrix representation of this generator is given by:

$$G := \lim_{c\to\infty} K = \begin{pmatrix} 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.11}$$

The commutation relations of the generators are now given by:

$$[G, H] = iP \qquad [G, P] = 0 \qquad [H, P] = 0,$$

and a general element $\mathfrak{E}$ of the inhomogeneous Euclidean algebra is:

$$\mathfrak{E} = vG + \tfrac{1}{\hbar}(xP + tE).$$

The factor of $\hbar$ is introduced by convention in order that the dimensions of x be given by m, P by $kgms^{-1}$, t by s and E by kgm^2s^{-2} or Joules, while the general element of the algebra remains dimensionless. It is well known that this Lie algebra admits a non-trivial central extension, with central charge M [3, 51, 79]. The commutation relations of the central extension are:

$$[G, H] = iP \qquad [G, P] = iM \qquad [H, P] = 0. \tag{4.12}$$

The relative velocity v of the respective reference frames has dimensions of ms^{-1}, and so G has dimensions of $m^{-1}s$. Therefore, the central charge M has dimensions of mass - kg. This is the Lie algebra of the Galilean group in 1 (spatial) dimension, $\mathcal{G}(1)$. Notably, the matrix group $\Gamma(v, a_i)$ defined in Equation 4.7b is not actually a representation of the Poincaré group, nor is the matrix group in Equation 4.10 a representation of the Galilean group. Rather, the Poincaré group is the universal covering group (the topological central extension) of the matrix group defined in Equation 4.8 (cf. Sections 2.7.2 and 2.8). The Galilean group has the added complexity that it admits an algebraic central extension, and so the Galilean group is properly defined as the universal covering group of the group which has corresponding Lie algebra given in Equation 4.12.

İnönü and Wigner in [40] go on to describe the contractions for the unitary irreducible representations of the Poincaré group in $(1+3)$ dimensional spacetime, but there is no need to reproduce this here. In $(1+1)$ dimensions the subgroup S, with respect to which the contraction is carried out, is the group of translations in time. In $(1+3)$ dimensions, S is given by the group of translations in time together with spatial rotations in 3 dimensions. The resulting contracted group is the inhomogeneous Euclidean group in 3 dimensions: $\mathcal{IE}(3) \cong \mathcal{E}(3) \ltimes \mathcal{T}(4)$, which can be centrally extended to the Galilean group in 3 dimensions, $\mathcal{G}(3)$.

4.1.2.1 Interpretation

The Poincaré group describes the mechanics of two reference frames which are moving at a high relative constant (inertial) velocity with respect to one another; the group of special relativity. The group has a fundamental constant, the speed

of light c, which is a dimensional ratio between units of position and units of time. Consider three observers $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$, with observers $\mathcal{O}$ and $\mathcal{O}'$ moving at a relative velocity v and observers $\mathcal{O}'$ and $\mathcal{O}''$ moving at a relative velocity v'. Then observers $\mathcal{O}$ and $\mathcal{O}''$ are moving at a relative velocity v'' which is determined from v and v' by:

$$v'' = \frac{v + v'}{1 + \frac{vv'}{c^2}}. \tag{4.13}$$

This formula is easily determined from the multiplication of the matrices $\Gamma(v'') = \Gamma(v)\Gamma(v')$, with $\Gamma(v)$ defined in Equation 4.8. Poincaré transformations conserve the special relativistic line element $ds^2 = -dt^2 + \frac{1}{c^2}dx^2$ (or $d\mu^2 = -\frac{1}{c^2}de^2 + dp^2$ if the inhomogeneous group is taken to be translations in momentum and energy as opposed to translations in space and time). Additionally, if the Poincaré group is defined over the extended 1-phase space then the symplectic form $-de \wedge dt + dp \wedge dx$ remains invariant. In other words, there are two Poincaré invariant metrics, η and ζ defined (up to dimensions) by:

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \zeta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{4.14}$$

such that $dx^T \eta dx = (dx')^T \eta dx'$ and $dz^T \zeta dz = (dz')^T \zeta dz'$ where:

$$dz = \begin{pmatrix} dt \\ dx \\ dp \\ de \end{pmatrix}. \tag{4.15}$$

Taking the limit as $c \to \infty$ is equivalent to taking the limit of $\frac{v}{c} \to 0$. In other words, $v \ll c$. Under this limit, velocities add linearly:

$$v'' = v + v'. \tag{4.16}$$

This group, the Galilean group, still describes the physics of two reference frames which are inertial with respect to one another and move at a relative velocity v. However, the velocities v are necessarily much smaller than the speed of light. This is the group of classical inertial mechanics. Galilean transformations also preserve the symplectic metric. In addition, they separately preserve the Euclidean distance between two points dx^2 and the classical line element dt^2. In other words, under Galilean transformations the Euclidean metric η_E and the time metric η_c are both separately invariant, where:

$$\eta_E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \eta_c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.17}$$

4.1.3 The Quaplectic Group Contracts to the Hamilton Group

The homogeneous subgroup of the quaplectic group can be realised by the matrices $\Xi(v, f, r)$ of Equation 3.5 for $(1 + 1)$ dimensions. Using the form of the transformation matrices Ξ as expressed in Equation 3.11, the limits of the transformation matrix as $b, c \to \infty$ yields:

$$\Phi(v, f, r) := \lim_{b,c\to\infty} \Xi(v, f, r) \tag{4.18a}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v & 1 & 0 & 0 \\ f & 0 & 1 & 0 \\ r & -f & v & 1 \end{pmatrix}. \tag{4.18b}$$

This is actually the limit of the $\mathcal{SU}(1, 1)$ transformation matrix rather than the $\mathcal{U}(1, 1)$ transformation matrix, but the net effect of including the $\mathcal{U}(1)$ matrix of Equation 3.12 would be for the bottom left entry in the contracted matrix in Equation 4.18b to be ra rather than simply r, and this has no physical significance. The product of two elements in this set is given by:

$$\Phi(v_1, f_1, r_1)\Phi(v_2, f_2, r_2) = \Phi(v_1 + v_2, f_1 + f_2, r_1 + r_2 - f_1 v_2 + v_1 f_2)$$

and the inverse is given by:

$$\Phi^{-1}(v, f, r) = \Phi(-v, -f, -r).$$

Therefore the set of $\Phi(v, f, r)$ forms a group; the Hamilton group in 1 (spatial) dimension $\mathcal{H}a(1)$ (cf. [61, 59]). From Equation 3.11 one can see that this group operates on the phase space dz of Equation 4.15. The phase space is transformed as:

$$dt' = dt \tag{4.19a}$$

$$dx' = vdt + dx \tag{4.19b}$$

$$dp' = fdt + dp \tag{4.19c}$$

$$de' = rdt - fdx + vdp + de. \tag{4.19d}$$

This group has Lie algebra generators which are the contracted generators of Equations 3.14 with commutation relations defined in Equations 3.15. The contracted generators are:

$$G := \lim_{b,c\to\infty} K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix} \tag{4.20a}$$

$$F := \lim_{b,c\to\infty} N = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \tag{4.20b}$$

$$U := \lim_{b,c\to\infty} R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \tag{4.20c}$$

such that a general element $\mathfrak{A}$ of the algebra is given by:

$$\mathfrak{A} = vG + fF + rU. \tag{4.21}$$

The contraction of the A generator in Equation 3.14 is redundant since it is equal to the contraction of the R generator. These generators obey the following commutation relations:

$$[G, F] = 2iU \qquad [G, U] = 0 \qquad [F, U] = 0. \tag{4.22}$$

4.1.3.1 Interpretation

Equation 4.19a shows that time is invariant under Hamilton transformations, and therefore forms a Hamilton-invariant subspace. This means that the Equations 4.19 can be divided by dt to obtain:

$$\frac{dx'}{dt} = v + \frac{dx}{dt} \tag{4.23a}$$

$$\frac{dp'}{dt} = f + \frac{dp}{dt} \tag{4.23b}$$

$$\frac{de'}{dt} = r - f\frac{dx}{dt} + v\frac{dp}{dt} + \frac{de}{dt}. \tag{4.23c}$$

This leads to the identification of v with position evolution or velocity, f with momentum evolution or force and r with energy evolution or power. By making the identification:

$$dH(x, p, t) := rdt - fdx + vdp$$

such that

$$de' = de + dH(x, p, t),$$

one can derive Hamilton's equations:

$$\frac{\partial H}{\partial x} = -\dot{p} \qquad \frac{\partial H}{\partial p} = \dot{x}$$

where $\dot{p} \equiv f$ and $\dot{x} \equiv v$. Also,

$$\frac{\partial H}{\partial t} = r.$$

Consider three observers: $\mathcal{O}$, $\mathcal{O}'$ and $\mathcal{O}''$. Suppose that observers $\mathcal{O}$ and $\mathcal{O}'$ are moving[2] at a relative velocity v, force f and power r and observers $\mathcal{O}'$ and $\mathcal{O}''$

[2]See Section 3.1 for the definition of what it means to "move" at a relative velocity, force and power.

are moving at a relative velocity v', force f' and power r', then according to the matrix realisation in Equation 4.18b, the velocity, force and power of $\mathcal{O}''$ relative to $\mathcal{O}$ is given by:

$$\begin{aligned} v'' &= v' + v \\ f'' &= f' + f \\ r'' &= r' + r + vf' - fv'. \end{aligned}$$

The transformation matrix of Equation 4.18b and the transformations of Equations 4.19 describe the physics of two observers moving at a relative velocity v, force f and power r. The contraction effected by taking the limit of $c \to \infty$ means that the relative velocities in question are insignificant when compared to the speed of light, and hence the observers are non-relativistic (with respect to one another). The contraction effected by taking the limit of $b \to \infty$ means that the relative forces in question are insignificant when compared to the fundamental constant b. So this is the (homogeneous) relativity group of classical inertial mechanics. This group, along with the various forms of the inhomogeneous Hamilton group studied in 4.1.3.2, is due to Low (for example, [61, 59]).

4.1.3.2 The Inhomogeneous Hamilton Group

Nothing has yet been said about the normal subgroup. Indeed, in taking the limits $b, c \to \infty$ in the quaplectic group $\mathcal{Q}(1,1)$, the normal subgroup $\mathcal{H}(2)$ is not affected at all. However since the central extension of the quaplectic group has not been used in the analysis, the normal subgroup can be taken to be the translation group $\mathcal{T}(4)$. The (classical) inhomogeneous Hamilton Group $\mathcal{IH}a(1)$ is then defined to be:

$$\mathcal{IH}a(1) = \mathcal{H}a(1) \ltimes \mathcal{T}(4)$$

or in general [59]

$$\mathcal{IH}a(n) = \mathcal{H}a(n) \ltimes \mathcal{T}(2n+2).$$

A general element $\mathfrak{A}$ of the Lie algebra of this group is given by (cf. 4.21):

$$\mathfrak{A} = vG + fF + rU + \tfrac{1}{\hbar}(tE + xP + pX + eT). \tag{4.24}$$

The $\{E, P^i, X^i, T\}$ generators are the generators of the $\mathcal{T}(2n+2)$ normal subgroup. For $n = 1$, they can be represented by the 5×5 matrices of Equations 4.6. The 4×4 representations of $\{G, F, U\}$ will therefore be embedded in 5×5 matrices, by inserting a fifth row and column of zeros, in order to determine the commutation relations of the Lie algebra. Direct matrix calculation could be used to calculate the commutators or the limits $b, c \to \infty$ (and $\hbar \to 0$) can be taken directly in the commutation relations of Equations 3.17. The commutation relations of the inhomogeneous Hamilton group, in addition to those given in Equations 4.22, are

given by:

$$[G,T]=0 \qquad [G,X]=iT \qquad [G,P]=0 \qquad [G,E]=iP \tag{4.25a}$$
$$[F,T]=0 \qquad [F,X]=0 \qquad [F,P]=-iT \qquad [F,E]=iX \tag{4.25b}$$
$$[U,T]=0 \qquad [U,X]=0 \qquad [U,P]=0 \qquad [U,E]=iT, \tag{4.25c}$$

with all generators of the abelian normal subgroup mutually commuting. By abuse of notation, the generators $\{T,X,P,E\}$ of Equations 4.25 are not equivalent to the respectively labelled generators of Equations 3.17. Rather, they will correspond to these generators once the central extension of the translation group has been taken (cf. Equations 2.66). These commutation relations clearly have a central generator T. A general element of the group can be realised by the following matrix:

$$H(v,f,r,t,x,p,e)=\begin{pmatrix} 1 & 0 & 0 & 0 & t \\ v & 1 & 0 & 0 & x \\ f & 0 & 1 & 0 & p \\ r & -f & v & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.26}$$

The central extension of the inhomogeneous Hamilton group is the semidirect product of the central extension $\widehat{\mathcal{H}a}(1)$ of $\mathcal{H}a(1)$ and $\mathcal{H}(2)$ (or in general, it is the semidirect product of $\widehat{\mathcal{H}a}(n)$ and $\mathcal{H}(2n)$). The Hamilton group $\mathcal{H}a(1)$ admits an algebraic central extension [59]. The commutation relations of Equations 4.25 become:

$$[G,T]=0 \qquad [G,X]=iT \qquad [G,P]=iM \qquad [G,E]=iP \tag{4.27a}$$
$$[F,T]=0 \qquad [F,X]=iA \qquad [F,P]=-iT \qquad [F,E]=iX \tag{4.27b}$$
$$[U,T]=0 \qquad [U,X]=0 \qquad [U,P]=0 \qquad [U,E]=iT \tag{4.27c}$$
$$[T,E]=-iI \qquad [X,P]=iI. \tag{4.27d}$$

The Lie algebra has four central elements: T, I, M and A. Similar to the Galilean group, the central charge M is mass. The central charge I has units of action and is the central charge of the Heisenberg group. The central charge A, which has units of reciprocal tension, is a new central charge [59]. A general element $\mathfrak{A}$ of the algebra is given by:

$$\mathfrak{A}=vG+fF+rU+\tfrac{1}{\hbar}(tE+xP+pX+eT)+aA+mM+\iota I,$$

so that the group parameters a, m, and ι have units of tension, inverse mass and inverse action respectively. This group describes the physics of reference frames which are non-relativistic, non-strongly-interacting and non-inertial with respect to one another; with a quantum mechanical structure. The presence of the central charge I allows the Heisenberg group to be a subgroup of the inhomogeneous Hamilton group. Similarly, the central charge M allows the Galilean group to be a subgroup of the inhomogeneous group. This subgroup is described in Section 4.1.3.3.

4.1.3.3 Inertial Subgroup: The Galilean Group

The group described by the matrix realisation of Equation 4.26 has an obvious subgroup: the Galilean group in 1 dimension. By setting $f = r = 0$, the matrix becomes:

$$\mathcal{G}(v,t,x,p,e) = \begin{pmatrix} 1 & 0 & 0 & 0 & t \\ v & 1 & 0 & 0 & x \\ 0 & 0 & 1 & 0 & p \\ 0 & 0 & v & 1 & e \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.28}$$

This is a double-version of the Galilean group (cf. Equation 4.10). It clearly involves two subgroups: one which acts on space and time and the other which acts on momentum and energy. The central extension of the subgroup which acts on space and time is the Galilean group (cf. Section 4.1.2). It has generators E, P, X and T defined in Equations 4.6 as well as a doubled version of the generator G defined in Equation 4.11. This doubled generator, denoted $\widetilde{G}$, is given by:

$$\widetilde{G} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore this doubled Galilean group has the Lie algebra of Equations 4.12 (with $E \equiv H$) as well as the auxiliary commutation relations:

$$[G, X] = iT \qquad [G, T] = 0.$$

This "second version" of the Galilean group does not admit an algebraic central extension, and so it is probably more accurate to call it a second or additional inhomogeneous Euclidean group rather than a second Galilean group.

4.1.4 The Quaplectic Group Contracts to the Poincaré Group

The limit as $b \to \infty$ with $f = r = 0$ of the homogeneous transformation matrices $\Xi(v, f, r)$ of Equation 3.11 yields transformation matrices $\Lambda(v)$:

$$\begin{aligned} \Lambda(v) :&= \lim_{b\to\infty} \Xi(v,0,0) \\ &= \begin{pmatrix} 1 & \frac{v}{c^2} & 0 & 0 \\ v & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{v}{c^2} \\ 0 & 0 & v & 1 \end{pmatrix}. \end{aligned}$$

This is a double-version of the Lorentz transformation matrix in (1+1) dimensions. It has two subgroups: one which acts on space and time and the other which acts

on momentum and energy. The subgroup which acts on space and time is the Lorentz group in $(1+1)$ dimensions (cf. Section 2.7). As described in Section 2.8, the central extension of the semidirect product of the Lorentz group and the translation group is the Poincaré group. The homogeneous group only has one generator (in $(1+1)$ dimensions), which is the generator K defined in Equation 3.14a. The commutation relations for the Poincaré group in $(1+1)$ dimensions are then given by:

$$[K,T] = \frac{i}{c^2}X \qquad [K,X] = \frac{i}{c^2}E$$
$$[K,P] = \frac{i}{c^2}H \qquad [K,E] = iP,$$

with all other generators commuting. The commutation relations of the first line are those of the space-time subgroup, belonging to the usual Poincaré group (cf. Equation 2.91). Two Minkowski metrics can be defined for this doubled Lorentz group over the frame dz of Equation 4.15, η_x for the space-time subspace and η_p for the momentum-energy subspace:

$$\eta_x = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \eta_p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This group also leaves invariant the remaining metric in Equation 4.14; the symplectic metric.

It has already been seen in Section 4.1.2 that the Poincaré group contracts to the Galilean group in the limit $c \to \infty$. This means that, along with the route taken in Section 4.1.3.3 to arrive at the Galilean group by first taking the limits $b, c \to \infty$ and then setting $f = r = 0$, there is a second route to the Galilean group from taking contractions of the quaplectic group. This second route is defined by first taking the limit $b \to \infty$ and setting $f = r = 0$, and then taking the limit $c \to \infty$. In other words, the contractions commute.

A third route is also possible, whereby first the limit $c \to \infty$ is taken while setting $v = r = 0$. The group resulting from this contraction is not examined in this work, but might be worthy of further study. However, if the limit $b \to \infty$ is then taken, one is left with the following (homogeneous) transformation matrix $\Pi(f)$:

$$\Pi(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ f & 0 & 1 & 0 \\ 0 & -f & 0 & 1 \end{pmatrix}.$$

The central extension of the semidirect product of this group and an appropriate translation group is not strictly a Galilean group since the physical parameters do not match that of the Galilean group, usually defined. However, it still has the same structure of a (double) Galilean group. There are two subgroups again, but

this time the first subgroup acts on momentum and time and the second subgroup acts on space and energy. It is useful to concentrate on the subgroup which acts on time and momentum, in order to understand the physical significance of this transformation group. Since $v = r = 0$, this describes a set of non-inertial reference frames which are stationary with respect to one another. It has (inhomogeneous) transformations:

$$\begin{aligned} t \to t'; \qquad & t' = t' + a_0 \\ p \to p'; \qquad & p' = p + ft + a_2, \end{aligned}$$

where a_0 is a translation in time and a_2 is a translation in momentum.

Chapter 5

Branching Rules

This chapter refers several times to a book on special functions written by Wilhelm Magnus, Fritz Oberhettinger and Raj Pol Soni [68]. It will be referred to as "Magnus".

In Section 5.1, several operators are redefined by the specific case of the quaplectic group $\mathcal{Q}(2)$ acting on a 2 dimensional Euclidean space. In Section 5.2 the branching problem is set up, for a basis in which the $\mathcal{O}(2)$ generator M is diagonal, with basis states $|r + 2m, r\rangle$ for fixed values of $\Delta := 2m$. There are three cases for m: negative, positive or zero. In Section 5.3, the case $m > 0$ is taken up and studied in detail. The coefficients are then determined and the action checked for the P_+ and P_- operators. In Section 5.3.1, the wavefunctions are rewritten in polar coordinates and the action of a_+ is verified. In Section 5.3.2 the momentum wavefunction is reconstructed in the position representation. In Section 5.3.3 the $P_\pm$ matrix elements are calculated which gives a first check to the wavefunction. The $X_\pm$ are then evaluated in Section 5.3.4. This allows the check of the Heisenberg algebra, which is done in Section 5.3.5. In Section 5.3.6 the operators $X_\pm^2$ and X_+X_- are evaluated. Finally, in Section 5.3.7, the quaplectic generators $Z'_\pm$ and U are evaluated.

5.1 Branching Notation

The quaplectic group in $(0 + 2)$ dimensions is given by:

$$\mathcal{Q}(2) = \mathcal{U}(2) \ltimes \mathcal{H}(2).$$

The Euclidean group in 2 dimensions is given by:

$$\mathcal{E}(2) = \mathcal{O}(2) \ltimes \mathcal{T}(2).$$

The Lie algebra of $\mathcal{U}(2)$ has generators $E^\mu{}_\nu$ and the Lie algebra of $\mathcal{H}(2)$ has generators $\overline{Z}^\mu$ and the conjugate Z_ν. These generators obey the commutation relations of Equations 3.20 and 3.21. As in Equation 3.24, auxiliary generators $W^\mu{}_\nu$ (represented by $S^\mu{}_\nu$) can be defined, with commutation relations given in Equations 3.28. Since there is no time-like component, the Greek indices will be changed to Roman letters such that the generators are $E^i{}_j$, $\overline{Z}^i$, Z_j and $S^i{}_j$ with

Group	Representation	Notation	Description
$\mathcal{T}(2)$, $\mathcal{H}(2)$	$\xi'(P_i)$, $\xi'(X_i)$	P_i, X_i, I	Translation/Heisenberg generators
$\mathcal{O}(2)$	$\rho'(J_{12})$	M	Rotation generator
$\mathcal{H}(2)$	$\xi'(Z_i)$, $\xi'(\overline{Z}_j)$	Z_i, $\overline{Z}^j$, I	Z-type Heisenberg generators
$\mathcal{U}(2)$	$\rho'(E^i{}_j)$	$Z^i{}_j$	$= \frac{1}{2}\left\{\overline{Z}^i, Z_j\right\}$
	$\sigma'(E^i{}_j)$	$S^i{}_j$	External "spin"
$\mathcal{Q}(2)$	$\tau'(E^i{}_j)$	$E^i{}_j$	$:= \sigma'(E^i{}_j) + \rho'(E^i{}_j) = S^i{}_j + Z^i{}_j$
	$\tau'(Z_i)$, $\tau'(\overline{Z}_j)$	Z_i, $\overline{Z}^j$, I	$\mathcal{H}(2)$ generators in $\mathcal{Q}(2)$ rep

Table 5.1: Consolidated Notation for the $\mathcal{Q}(2) \supset \mathcal{E}(2)$ Reduction.

$i, j = 1, 2$. The generators and the representations of the generators with the notation to be used for those representations in this chapter are summarised in Table 5.1.

Recalling that $Z^i{}_j$ is defined as an anticommutator of Heisenberg generators:

$$Z^i{}_j := \tfrac{1}{2}\left\{\overline{Z}^i, Z_j\right\},$$

the trace of this generator can be written in terms of the number operator $\widehat{N} := \overline{Z}^i Z_i$:

$$\begin{aligned} Z^i{}_i &= \tfrac{1}{2}(\overline{Z}^i Z_i + Z_i \overline{Z}^i) && \text{(5.1a)} \\ &= \tfrac{1}{2}(2\widehat{N} - \left[\overline{Z}^i, Z_i\right]) && \text{(5.1b)} \\ &= \widehat{N} + \tfrac{1}{2}n && \text{(5.1c)} \\ &= \widehat{N} + 1. && \text{(5.1d)} \end{aligned}$$

Additionally,

$$Z^i{}_i = \tfrac{1}{2}(X^2 + P^2), \qquad \text{(5.1e)}$$

from Equation 3.18. In [74], Shamaly denotes the operator $X^2 + P^2$ by the letter G. Shamaly then derives all of the Casimir invariants D_n, $n = 1, ..., 4$ of $\mathcal{U}(1, 3)$ in terms of this operator, as follows:

$$\begin{aligned} D_1 = Z^\mu{}_\mu &= \tfrac{1}{2}G && \text{(5.2a)} \\ D_2 = Z^\mu{}_\nu Z^\nu{}_\mu &= \tfrac{1}{4}(G^2 - 12) && \text{(5.2b)} \\ D_3 = Z^\mu{}_\nu Z^\nu{}_\rho Z^\rho{}_\mu &= \tfrac{1}{8}(G^3 + 3G^2 - 15G - 48) && \text{(5.2c)} \\ D_4 = Z^\mu{}_\nu Z^\nu{}_\rho Z^\rho{}_\sigma Z^\sigma{}_\mu &= \tfrac{1}{16}(G^4 + 6G^3 - 6G^2 + 96G - 156). && \text{(5.2d)} \end{aligned}$$

It will be useful to note that for the general (p, q) dimensional case, where $n = p+q$, the Casimir D_2 can be written as:

$$D_2 = (\widehat{N} + \tfrac{1}{2}n)^2 - \tfrac{1}{4}(n^2 - n). \qquad \text{(5.3)}$$

The first two $\mathcal{U}(2)$ Casimir operators are given by:

$$U = E^i{}_i \tag{5.4}$$

(the $\mathcal{U}(1)$ generator) from Equation 3.22 and

$$V = E^i{}_j E^j{}_i. \tag{5.5}$$

These will be used to label states. Additionally, the generators of $\mathcal{SU}(2)$ can be written in terms of the generators $E^i{}_j$ and U as follows:

$$A^i{}_j = E^i{}_j - \tfrac{1}{2}\delta^i{}_j U.$$

Rearranging this equation allows the Casimir V from Equation 5.5 to be written as:

$$\begin{aligned} V &= A^i{}_j A^j{}_i + \tfrac{1}{2}U^2 && \text{(5.6a)} \\ &= 2J(J+1) + \tfrac{1}{2}U^2, && \text{(5.6b)} \end{aligned}$$

where $J = 0, \frac{1}{2}, 1, \frac{3}{2}, ...$ is the eigenvalue of the $\mathcal{SU}(2)$ Casimir operator $\mathbf{J}^2$.

In the reduction problem the $\mathcal{Q}(2)$ states will be labelled by $\mathcal{E}(2) = \mathcal{O}(2) \ltimes \mathcal{T}(2)$ states. The $\mathcal{O}(2)$ group is generated by $M := S_y + Z_y$, where $S_y = -i\frac{1}{2}(S_{12} - S_{21})$ and $Z_y = -\frac{1}{2}i(Z_{12} - Z_{21})$. The $\mathcal{T}(2)$ group has generators P_1 and P_2 (translations).

Only the scalar case will be considered. In this case, take $\varepsilon^i{}_j = \epsilon\delta^i{}_j$. The generator U is given by (from Equations 3.27, 5.1d and 5.4):

$$\begin{aligned} U &= E^i{}_i \\ &= \varepsilon^i{}_i + Z^i{}_i \\ &= 2\epsilon + \widehat{N} + 1. \end{aligned}$$

From Equation 5.5, the Casimir V is given by:

$$\begin{aligned} V &= \varepsilon^i{}_j \varepsilon^j{}_i + 2\varepsilon^i{}_j Z^j{}_i + Z^i{}_j Z^j{}_i \\ &= \varepsilon \cdot \varepsilon + 2\varepsilon \cdot Z + Z \cdot Z. \end{aligned}$$

The first term is given by:

$$\varepsilon \cdot \varepsilon = 2\epsilon^2;$$

the second term is given by:

$$2\varepsilon \cdot Z = 2\epsilon(\widehat{N} + 1);$$

And the third term is given by, from Equation 5.3:

$$Z \cdot Z = (\widehat{N} + 1)^2 - \tfrac{1}{2}.$$

Combining the three terms gives V as:

$$V = 2\epsilon^2 + 2\epsilon(\widehat{N} + 1) + (\widehat{N} + 1)^2 - \tfrac{1}{2},$$

but, from Equation 5.6b, $V = 2J(J+1) + \frac{1}{2}U^2$. Therefore, using $U = 2\epsilon + \widehat{N} + 1$:

$$\begin{aligned} 2J(J+1) + \tfrac{1}{2}(2\epsilon + \widehat{N} + 1)^2 &= 2\epsilon^2 + 2\epsilon(\widehat{N}+1) + (\widehat{N}+1)^2 - \tfrac{1}{2} \\ \Rightarrow 2J(J+1) + 2\epsilon^2 + 2\epsilon\widehat{N} + 2\epsilon + \tfrac{1}{2}\widehat{N}^2 + \widehat{N} + \tfrac{1}{2} &= 2\epsilon^2 + 2\epsilon\widehat{N} + 2\epsilon + \widehat{N}^2 + 2\widehat{N} + \tfrac{1}{2} \\ \Rightarrow 2J(J+1) &= \tfrac{1}{2}\widehat{N}^2 + \widehat{N} \\ &= \widehat{N}(\tfrac{1}{2}\widehat{N} + 1). \\ \therefore J &= \tfrac{1}{2}\widehat{N} \\ \text{or} \quad J &= -\tfrac{1}{2}\widehat{N} - 1. \end{aligned}$$

J cannot be negative, and so the second root can be discarded. This leaves the only solution as $J = \frac{1}{2}\widehat{N}$.

5.2 Branching: $\mathcal{Q}(2) \supset \mathcal{E}(2)$

The Heisenberg generators for $\mathcal{Q}(2)$ can be written in terms of creation and annihilation operators as follows:

$$\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad \begin{pmatrix} \overline{Z}^1 \\ \overline{Z}^2 \end{pmatrix} = \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}, \tag{5.7}$$

with

$$\left[a^\dagger, a\right] = \left[b^\dagger, b\right] = 1. \tag{5.8}$$

The other combinations of operators all mutually commute. There are four $Z^i{}_j$ generators. These will be renamed as $Z_+ \equiv Z^1{}_2$, $Z_- \equiv Z^2{}_1$ and the final two will be combined in $Z_z \equiv \frac{1}{2}(Z^1{}_1 - Z^2{}_2)$. Then:

$$Z_+ = a^\dagger b \tag{5.9a}$$
$$Z_- = b^\dagger a \tag{5.9b}$$
$$Z_z = \tfrac{1}{2}(a^\dagger a - b^\dagger b), \tag{5.9c}$$

with, of course,

$$\begin{aligned} Z^i{}_i &= Z^1{}_1 + Z^2{}_2 \\ &= \widehat{N} + 1 \\ &= a^\dagger a + b^\dagger b + 1. \end{aligned}$$

The generators $\{Z_+, Z_-, Z_z\}$ have the following commutation relations:

$$\begin{aligned} [Z_z, Z_\pm] &= \pm Z_\pm \\ [Z_+, Z_-] &= 2Z_z. \end{aligned}$$

Similarly, new "circular oscillator" creation and annihilation operators can be defined as:

$$a_{\pm} := \frac{1}{\sqrt{2}}(a \mp ib) \tag{5.10a}$$

$$a_{\pm}^{\dagger} := \frac{1}{\sqrt{2}}(a^{\dagger} \mp ib^{\dagger}). \tag{5.10b}$$

This allows new momentum and position operators to be defined in terms of these new creation and annihilation operators as (cf. Equation 3.18):

$$\begin{aligned} P_+ &= i(a_+^{\dagger} - a_-) & \qquad P_- &= i(a_-^{\dagger} - a_+) \\ X_+ &= (a_+^{\dagger} + a_-) & \qquad X_- &= (a_-^{\dagger} + a_+), \end{aligned}$$

with

$$[X_+, P_-] = [X_-, P_+] = 2i.$$

Then:

$$\begin{aligned} P_+P_- &= (P_1)^2 + (P_2)^2 \\ &= (a_+^{\dagger}a_+ + a_-^{\dagger}a_- + 1 - a_+^{\dagger}a_-^{\dagger} - a_+a_-) \\ X_+X_- &= (X_1)^2 + (X_2)^2 \\ &= (a_+^{\dagger}a_+ + a_-^{\dagger}a_- + 1 + a_+^{\dagger}a_-^{\dagger} + a_+a_-) \end{aligned}$$

The $\mathcal{U}(1)$ generator U can be written in terms of these operators as:

$$\begin{aligned} U &= 2\epsilon + \tfrac{1}{2}(X_+X_- + P_+P_-) \\ &= 2\epsilon + a_+^{\dagger}a_+ + a_-^{\dagger}a_- + 1. \end{aligned} \tag{5.11}$$

The second $\mathcal{Q}(2)$ Casimir C_1 is then:

$$C_1 = UI - \tfrac{1}{2}(X_+X_- + P_+P_-).$$

Also, a new set of Heisenberg generators $Z'_{\pm}$ and Z_y can be defined as:

$$Z'_+ = a_+^{\dagger}a_- = \tfrac{1}{2}(X_+^2 + P_+^2) \tag{5.12a}$$

$$Z'_- = a_-^{\dagger}a_+ = \tfrac{1}{2}(X_-^2 + P_-^2) \tag{5.12b}$$

$$Z_y = \tfrac{1}{2}(a_+^{\dagger}a_+ - a_-^{\dagger}a_-). \tag{5.12c}$$

The diagonalisation of $\mathcal{E}(2)$ representations can now proceed in a basis where the $\mathcal{O}(2)$ generator M is diagonal. States $|r_+, r_-\rangle$ can be written in terms of the creation and annihilation operators of Equation 5.10 as:

$$|r_+, r_-\rangle := \frac{(a_+^{\dagger})^{r_+}(a_-^{\dagger})^{r_-}}{\sqrt{r_+!r_-!}} |0, 0\rangle .$$

The creation and annihilation operators act in the usual manner on their respective + or − modes:

$$a_+^\dagger |r_+, r_-\rangle = \sqrt{r_+ + 1}\, |r_+ + 1, r_-\rangle ; \qquad a_+ |r_+, r_-\rangle = \sqrt{r_+}\, |r_+ - 1, r_-\rangle$$
$$a_-^\dagger |r_+, r_-\rangle = \sqrt{r_- + 1}\, |r_+, r_- + 1\rangle ; \qquad a_- |r_+, r_-\rangle = \sqrt{r_-}\, |r_+, r_- - 1\rangle .$$

In the scalar case ($S_y = 0$), the generator M is given by:

$$\begin{aligned} M &:= Z_y \\ &= \tfrac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-). \end{aligned}$$

In the general spin-S case, M has eigenvalues m:

$$\begin{aligned} M(|r_+, r_-\rangle \otimes |s, m_s\rangle) &= (\tfrac{1}{2}(a_+^\dagger a_+ - a_-^\dagger a_-))(|r_+, r_-\rangle \otimes |s, m_s\rangle) \\ &= (\tfrac{1}{2}(r_+ - r_-) + m_s)(|r_+, r_-\rangle \otimes |s, m_s\rangle) \\ &= m(|r_+, r_-\rangle \otimes |s, m_s\rangle), \end{aligned}$$

such that $m = \frac{1}{2}(r_+ - r_-) + m_s$. In the scalar case, with $s = 0$:

$$\begin{aligned} m &= \tfrac{1}{2}\Delta \\ &\equiv \tfrac{1}{2}(r_+ - r_-) \end{aligned}$$

where $\Delta = (r_+ - r_-)$ by definition. By fixing values of Δ such that:

$$\begin{aligned} r := r_- &= r_+ - 2m && (m > 0) \\ r := r_+ &= r_- && (m = 0) \\ r := r_+ &= r_- + 2m && (m < 0), \end{aligned}$$

basis states can be set up for various cases of m as follows:

$$|p^2; m\rangle := \begin{cases} \sum_{r=0}^{\infty} \lambda_r^+(p^2, m)\, |r + 2m, r\rangle & (m > 0) \\ \sum_{r=0}^{\infty} \lambda_r^0(p^2)\, |r, r\rangle & (m = 0) \\ \sum_{r=0}^{\infty} \lambda_r^-(p^2, m)\, |r, r - 2m\rangle & (m < 0) \end{cases} \tag{5.13}$$

where $\lambda_r^\pm$, and λ_r^0 are coefficients to be determined. This can be done by demanding that the states be eigenstates of the $\mathcal{O}(2) \ltimes \mathcal{T}(2)$ Casimir $P^2 = P_+ P_-$ and following the method of Lindblad and Nagel [52]. There are really only two cases since the case $m < 0$ is the same as $m > 0$. The second case can also be seen to be a particular form of the first case, where there is the added restriction that $m = 0$.

5.3 Branching: Case $m > 0$

For the case $m > 0$ and replacing λ_r^+ with λ_r for brevity of notation, the Casimir P^2 has eigenvalues p^2:

$$P^2 |p^2, m\rangle = \sum_{r=0}^{\infty} p^2 \lambda_r |r + 2m, r\rangle .$$

Acting on a state $|r+2m,r\rangle$, P^2 would have the following effect:

$$\begin{aligned}P^2\,|r+2m,r\rangle =&(a_+^\dagger a_+ + a_-^\dagger a_- + 1 - a_+^\dagger a_-^\dagger - a_+ a_-)\,|r+2m,r\rangle\\ =&(2r+2m+1)\,|r+2m,r\rangle -\\ &-\sqrt{(r+1)(r+2m+1)}\,|r+2m+1,r+1\rangle -\\ &-\sqrt{r(r+2m)}\,|r+2m-1,r-1\rangle\,.\end{aligned}$$

Applying this to the case $m>0$ results in:

$$\begin{aligned}\sum_{r=0}^{\infty} p^2\lambda_r(p^2,m)\,|r+2m,r\rangle =&\sum_{r=0}^{\infty}[(2r+2m+1)\lambda_r - \sqrt{r(r+2m)}\lambda_{r-1} -\\ &-\sqrt{(r+1)(r+2m+1)}\lambda_{r+1}]\,|r+2m,r\rangle\,,\end{aligned}$$

where the sums have been shifted in the second and third terms on the right hand side. Equating coefficients, this results in the following recurrence relation:

$$(p^2-2r-2m-1)\lambda_r + \sqrt{r(r+2m)}\lambda_{r-1} + \sqrt{(r+1)(r+2m+2)}\lambda_{r+1} = 0. \tag{5.14}$$

New coefficients μ_r can be introduced in order to simplify this recurrence relation. Define:

$$\lambda_r(p^2,m) \equiv N(p^2,m)\frac{\Gamma(r+2m+1)^{\frac{1}{2}}}{\Gamma(r+1)^{\frac{1}{2}}}\mu_r(p^2,m), \tag{5.15}$$

where $\Gamma(n+1)=n\Gamma(n)=n!$ and N is a normalisation constant. Then:

$$\begin{aligned}\sqrt{r(r+2m)}\lambda_{r-1} =&Nr^{\frac{1}{2}}(r+2m)^{\frac{1}{2}}\frac{\Gamma(r+2m)^{\frac{1}{2}}}{\Gamma(r)^{\frac{1}{2}}}\mu_{r-1} &\text{(5.16a)}\\ =&Nr\frac{\Gamma(r+2m+1)^{\frac{1}{2}}}{\Gamma(r+1)^{\frac{1}{2}}}\mu_{r-1} &\text{(5.16b)}\end{aligned}$$

and

$$\begin{aligned}\sqrt{(r+1)(r+2m+1)}\lambda_{r+1} =&N(r+1)^{\frac{1}{2}}(r+2m+1)^{\frac{1}{2}}\frac{\Gamma(r+2m+2)^{\frac{1}{2}}}{\Gamma(r+2)^{\frac{1}{2}}}\mu_{r+1} &\text{(5.16c)}\\ =&N(r+2m+1)\frac{\Gamma(r+2m+1)^{\frac{1}{2}}}{\Gamma(r+1)^{\frac{1}{2}}}\mu_{r-1}. &\text{(5.16d)}\end{aligned}$$

Substituting Equations 5.16b and 5.16d back into the recurrence relation of Equation 5.14 and eliminating all common factors results in the new recurrence relation for μ_r:

$$(p^2-2r-2m-1)\mu_r + r\mu_{r-1} + (r+2m+1)\mu_{r+1} = 0.$$

For convenience, this recurrence relation can be rewritten as:

$$(p^2 - 2m - 1)\mu_r + (2m + 1)\mu_{r+1} - 2r\mu_r + r\mu_{r-1} + r\mu_{r+1} = 0. \tag{5.17}$$

Still following Lindblad and Nagel [52], this equation can be solved using Laplace's method. Define the ansatz:

$$\mu_r := \int_C t^{r-1}\mu(t; p^2, m)dt$$

where C is the contour of integration. This is, in effect, a generating function for μ_r. Then note that, from the product rule:

$$t^r \frac{d\mu}{dt} \to -rt^{r-1}\mu(t) \Rightarrow r\mu_r \to -t\frac{d\mu}{dt}t^{r-1}, \tag{5.18}$$

so long as certain boundary conditions are met. Additionally:

$$\begin{aligned}\mu_r &\to \mu(t)t^{r-1}\\ \mu_{r+1} &\to t\mu(t)t^{r-1}\\ \mu_{r-1} &\to \tfrac{1}{t}\mu(t)t^{r-1}\end{aligned}$$

and, from 5.18:

$$\begin{aligned}(r+1)\mu_{r+1} \to -t^2\frac{d\mu}{dt}t^{r-1}, \qquad &\text{or} \qquad r\mu_{r+1} \to (-t^2\frac{d\mu}{dt} - t\mu)t^{r-1};\\ \text{and } (r-1)\mu_{r-1} \to -t\frac{d\mu}{dt}t^{r-2} \qquad &\text{or} \qquad r\mu_{r-1} \to (-\frac{d\mu}{dt} + \frac{1}{t}\mu)t^{r-1}\end{aligned}$$

These can be substituted into Equation 5.17. Then, dividing by the common factor of t^{r-1}, Equation 5.17 becomes:

$$\begin{aligned}&(p^2 - 2m - 1)\mu(t)t^{r-1} + (2m+1)t\mu(t)t^{r-1} + 2t\frac{d\mu}{dt}t^{r-1} - \frac{d\mu}{dt}t^{r-1} +\\ &\qquad + \frac{1}{t}\mu(t)t^{r-1} - t^2\frac{d\mu}{dt}t^{r-1} - t\mu(t)t^{r-1} = 0\\ \Rightarrow \qquad &(p^2 - 2m - 1)\mu + (2m+1)t\mu + 2t\frac{d\mu}{dt} - \frac{d\mu}{dt} + \frac{1}{t}\mu - t^2\frac{d\mu}{dt} - t\mu = 0\\ \Rightarrow \qquad &(t-1)^2\frac{d\mu}{dt} - (p^2 - 2m + 2mt + \frac{1}{t} - 1)\mu = 0.\end{aligned}$$

This can be rearranged:

$$\frac{d\mu}{dt} = \left[\frac{p^2 - 2m - 1}{(t-1)^2} + \frac{2mt}{(t-1)^2} + \frac{1}{t(t-1)^2}\right]\mu$$

and integrated:

$$\frac{d}{dt}\ln\mu(t) = \frac{p^2 - 2m - 1}{(t-1)^2} + \frac{2mt}{(t-1)^2} + \frac{1}{t(t-1)^2}. \tag{5.19}$$

Using partial fractions,

$$\begin{aligned} 2m\frac{t}{(t-1)^2} &= 2m(\frac{a}{t-1} + \frac{b}{(t-1)^2}) \\ &= \frac{2m}{t-1} + \frac{2m}{(t-1)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{1}{t(t-1)^2} &= \frac{a}{t} + \frac{b}{t-1} + \frac{c}{(t-1)^2} \\ &= \frac{1}{t} - \frac{1}{t-1} + \frac{1}{(t-1)^2}, \end{aligned}$$

and so Equation 5.19 becomes:

$$\frac{d}{dt}\ln\mu(t) = \frac{p^2}{(t-1)^2} + \frac{2m-1}{t-1} + \frac{1}{t}.$$

This can now be integrated again, resulting in:

$$\begin{aligned} \ln\mu(t) &= -\frac{p^2}{t-1} + (2m-1)\ln(t-1) + \ln t + N \\ \Rightarrow \mu(t) &= N(p^2,m)t(t-1)^{2m-1}e^{\frac{-p^2}{t-1}}. \end{aligned}$$

This gives for μ_r^+ (returning to the + notation):

$$\mu_r^+ = N^+(p^2,m)\int t^r(t-1)^{2m-1}e^{\frac{-p^2}{t-1}}dt.$$

As noted above, there are certain constraints on the contour of integration C in order for this to be true. Indeed, C must be chosen either so that the integrand vanishes at the end points or that the integrand is over a closed contour with the integrand single-valued and non-divergent. A change of variable: $s := \frac{-1}{t-1}$, with $ds = (t-1)^{-2}dt = s^2dt$ leads to:

$$\mu_r^+ = N^+(p^2,m)\int e^{p^2s}s^{-(2m+1)}(1-\frac{1}{s})^r ds$$

This is in the same form as the following integral on page 277 of Magnus *et al* [68]:

$${}_1F_1(a;c;z) = \frac{\Gamma(c)z^{1-c}}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{zt}t^{-c}(1-\frac{1}{t})^{-a}dt,$$

where the real part of c is positive and $\gamma > 1$. By making the substitutions $a = -r$, $c = 2m+1$ and $z = p^2$, μ_r^+ becomes:

$$\mu_r^+ = N^+(p^2,m)\frac{2\pi i}{\Gamma(2m+1)(p^2)^{-2m}}\ {}_1F_1(-r;2m+1;p^2).$$

From [68] (noting that the subscript ν refers to a rising factorial or the "Pochhammer function"):

$$\begin{aligned} L_\nu^{(\alpha)} &= \frac{(\alpha+1)_\nu}{\Gamma(\nu+1)}\ {}_1F_1(-\nu;\alpha+1;z) \\ &= \frac{\Gamma(\alpha+\nu+1)}{\Gamma(\alpha+1)\Gamma(\nu+1)}\ {}_1F_1(-\nu;\alpha+1;z). \end{aligned}$$

This allows μ_r^+ to be written in terms of the associated Laguerre polynomials L_ν^α, with $\nu = r$, $\alpha = 2m$ and $z = p^2$:

$$\mu_r^+ = N^+(p^2,m)p^{4m}\frac{\Gamma(r+1)}{\Gamma(r+2m+1)}L_r^{2m}(p^2)$$

and therefore, recalling the definition of μ_r^+ in terms of λ_r^+, from Equation 5.15:

$$\lambda_r^+ = N^+(p^2,m)p^{4m}\left(\frac{\Gamma(r+1)}{\Gamma(r+2m+1)}\right)^{1/2}L_r^{2m}(p^2). \tag{5.20}$$

The dependence of the normalisation factor $N^+(p^2,m)$ on p^2 can now be determined by the action of the $P_\pm$ operators, since $P_\pm = p\,|p^2, m\pm\tfrac{1}{2}\rangle$. The operator P_+ can be written in terms of circular oscillators: $P_+ = i(a_+^\dagger - a_-)$. Therefore (with $N \equiv N^+(p^2,m)$ for brevity of notation):

$$\begin{aligned} i(a_+^\dagger - a_-)\,|p^2,m\rangle =& iN\sum_{r=0}^{\infty}p^{4m}\left(\frac{\Gamma(r+1)}{\Gamma(r+2m+1)}\right)^{1/2}L_r^{2m}\cdot \\ &\cdot\left[(r+2m+1)^{1/2}\,|r+2m+1,r\rangle - r^{1/2}\,|r+2m,r-1\rangle\right] \\ =& iN\sum_{r=0}^{\infty}p^{4m}\left(\frac{\Gamma(r+1)}{\Gamma(r+2m+2)}\right)^{1/2}\left[(r+2m+1)L_r^{2m} - \right. \\ &\left. -(r+1)L_{r+1}^{2m}\right]|r+2m+1,r\rangle \end{aligned}$$

where r has been replaced by $r+1$ in the second term. Magnus [68] has an occurrence relation for associated Laguerre polynomials of the form:

$$xL_n^{\alpha+1}(x) = (n+\alpha+1)L_n^\alpha(x) - (n+1)L_{n+1}^\alpha(x).$$

The equation therefore becomes:

$$P_+\,|p^2,m\rangle = iN^+(p^2,m)\sum_{r=0}^{\infty}p^{2m+2}\left(\frac{\Gamma(r+1)}{\Gamma(r+2m+2)}\right)^{1/2}L_r^{2m+1}\,|r+2m+1,r\rangle\,.$$

But,

$$p\,|p^2,m+\tfrac{1}{2}\rangle = N^+(p^2,m+\tfrac{1}{2})p^{4m+3}\left(\tfrac{\Gamma(r+1)}{\Gamma r+2m+2}\right)^{1/2}L_r^{2m}(p^2)\,|r+2m+1,r\rangle\,,$$

and so:

$$\frac{N^+(p^2,m)}{N^+(p^2,m+\frac{1}{2})} = -ip.$$

A similar calculation for P_- shows that:

$$\frac{N^+(p^2,m)}{N^+(p^2,m-\frac{1}{2})} = \frac{i}{p}.$$

A formula for $N^+(p^2,m)$ consistent with these two relations is:

$$N^+(p^2,m) \propto \left(\frac{i}{p}\right)^{2m}. \tag{5.21}$$

As can be easily checked, the action of the $\mathcal{O}(2)$ generator M simply returns its eigenvalue m:

$$M\,|p^2,m\rangle = m\,|p^2,m\rangle$$

as required.

5.3.1 Orthonormality and Check of a_+ Action

From [16][1], the wavefunctions of the oscillator states - in two dimensional polar coordinates ρ, θ (where $x = \rho\cos\theta$ and $y = \rho\sin\theta$) - are given by:

$$\begin{aligned}\langle\rho,\theta|r+2m,r\rangle &= C_r^{2m} e^{-\rho^2/2}\rho^{2m}\,{}_1F_1(-r;2m+1;\rho^2)e^{2im\theta}\\ &= C_r^{2m} e^{-\rho^2/2}\rho^{2m}\frac{\Gamma(2m+1)\Gamma(r+1)}{\Gamma(2m+r+1)}L_r^{2m}(\rho^2).\end{aligned}$$

With respect to the weighting function $(\rho^2)^{2m}e^{-\rho^2/2}$, two Laguerre polynomials $L_r^{2m}(\rho^2)$ and $L_{r'}^{2m}(\rho^2)$ are orthogonal:

$$\int_0^\infty (\rho^2)^{2m}e^{-\rho^2/2}L_r^{2m}(\rho^2)L_{r'}^{2m}(\rho^2)d\rho^2 = \frac{\Gamma(r+2m+1)}{\Gamma(r+1)}\delta_{rr'}.$$

For the wavefunctions to be orthonormal, it is required that:

$$\begin{aligned}1 &= 2\pi(C_r^{2m})^2\left(\frac{\Gamma(2m+1)\Gamma(r+1)}{\Gamma(2m+r+1)}\right)^2\int_0^\infty \frac{1}{2}e^{-\rho^2}(\rho^2)^{2m}(L_r^{2m}(\rho^2))^2d\rho^2\\ &= \pi(C_r^{2m})^2\left(\frac{\Gamma(2m+1)\Gamma(r+1)}{\Gamma(2m+r+1)}\right)^2\frac{\Gamma(r+2m+1)}{\Gamma(r+1)}\\ &= \pi(C_r^{2m})^2\Gamma(2m+1)^2\frac{\Gamma(r+1)}{\Gamma(2m+r+1)};\\ \therefore C_r^{2m} &= \frac{1}{\pi^{\frac{1}{2}}\Gamma(2m+1)}\left(\frac{\Gamma(2m+r+1)}{\Gamma(r+1)}\right)^{\frac{1}{2}}.\end{aligned}$$

[1]Specifically, Problem 7.20 on pages 373-374.

Therefore, the wavefunction becomes:

$$\langle \rho, \theta | r + 2m, r \rangle = \frac{\Phi_r^{2m}}{\pi^{\frac{1}{2}}} \left(\frac{\Gamma(r+1)}{\Gamma(2m+r+1)} \right)^{\frac{1}{2}} e^{-\rho^2/2} \rho^{2m} L_r^{2m}(\rho^2) \tag{5.22}$$

where Φ is a phase. This expression can be checked against the action of the circular creation and annihilation operators a_+ and $a_+^\dagger$, a_- and $a_-^\dagger$. The operator a_+ will be used as an example. From Equations 3.18 and 5.7:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}(X - iP_x) & a^\dagger &= \frac{1}{\sqrt{2}}(X + iP_x) \\ b &= \frac{1}{\sqrt{2}}(Y - iP_y) & b^\dagger &= \frac{1}{\sqrt{2}}(Y + iP_y) \\ X &= \frac{1}{\sqrt{2}}(a + a^\dagger) & P_x &= \frac{i}{\sqrt{2}}(a - a^\dagger) \\ Y &= \frac{1}{\sqrt{2}}(b + b^\dagger) & P_y &= \frac{i}{\sqrt{2}}(b - b^\dagger) \end{aligned}$$

Furthermore, from Equation 5.10:

$$\begin{aligned} a_+ &= \frac{1}{\sqrt{2}}(a - ib) \\ &= \tfrac{1}{2}(X - iP_x - iY - P_y) \\ &= \tfrac{1}{2}(x + \partial_x - iy - i\partial_y), \end{aligned} \tag{5.23}$$

where $P_x \to i\partial_x$ and $P_y \to i\partial_y$. Now, since $x = \rho\cos\theta$, $y = \rho\sin\theta$ $\rho^2 = x^2 + y^2$ and $y/x = \tan\theta$, the Jacobian matrix is given by:

$$\begin{aligned} \frac{\partial(x,y)}{\partial(\rho,\theta)} &= \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{pmatrix} \end{aligned}$$

This is easily inverted, giving:

$$\begin{aligned} \frac{\partial(\rho,\theta)}{\partial(x,y)} &= \left(\frac{\partial(x,y)}{\partial(\rho,\theta)} \right)^{-1} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{\rho}\sin\theta & \frac{1}{\rho}\cos\theta \end{pmatrix}. \end{aligned}$$

Therefore:

$$\begin{aligned} \frac{\partial \rho}{\partial x} &= \cos\theta & \frac{\partial \rho}{\partial y} &= \sin\theta \\ \frac{\partial \theta}{\partial x} &= -\frac{1}{\rho}\sin\theta & \frac{\partial \theta}{\partial y} &= \frac{1}{\rho}\cos\theta, \end{aligned}$$

and, from the chain rule:

$$\begin{aligned}\partial_x &= \frac{\partial\rho}{\partial x}\partial_\rho + \frac{\partial\theta}{\partial x}\partial_\theta \\ &= \cos\theta\partial_\rho - \frac{1}{\rho}\sin\theta\partial_\theta \\ \partial_y &= \frac{\partial\rho}{\partial y}\partial_\rho + \frac{\partial\theta}{\partial y}\partial_\theta \\ &= \sin\theta\partial_\rho + \frac{1}{\rho}\cos\theta\partial_\theta,\end{aligned}$$

and so therefore:

$$\partial_x - i\partial_y = e^{-i\theta}(\partial_\rho - \tfrac{i}{\rho}\partial_\theta).$$

The operator a_+ in Equation 5.23 can now be written as:

$$\begin{aligned}a_+ &= \frac{1}{2}[(x - iy) + (\partial_x - i\partial_y)] \\ &= \frac{1}{2}[\rho e^{-i\theta} + e^{-i\theta}(\partial_\rho - \frac{i}{\rho}\partial_\theta)]\end{aligned}$$

The operator a_+ is to act on functions of the form $f(\rho^2)e^{2mi\theta}$. Therefore:

$$\partial_\theta \to 2mi \qquad \rho\partial_\rho \to 2\rho^2\partial_{\rho^2},$$

and the operator becomes:

$$a_+ \to \frac{1}{2\rho}(\rho^2 + 2m + 2\rho^2\partial_\rho). \tag{5.24}$$

From [68][2], the derivative of an associated Laguerre polynomial is given by:

$$\begin{aligned}\frac{d}{dx}L_n^\alpha(x) &= -L_{n-1}^{\alpha+1}(x) \\ \Rightarrow \frac{\partial}{\partial\rho^2}L_r^{2m}(\rho^2) &= -L_{r-1}^{2m+1}(\rho^2).\end{aligned}$$

From the product rule, the differential part of a_+, $2\rho^2\partial_{\rho^2}$ results in three terms when acting on $|\psi\rangle := e^{-1/2\rho^2}\rho^{2m}L_r^{2m}(\rho^2)$ (using the shorthand notation $L \equiv L_r^{2m}(\rho^2)$):

$$\begin{aligned}&2\rho^2\partial_{\rho^2}(e^{-1/2\rho^2}\rho^{2m}L_r^{2m}(\rho^2)) = \\ &\quad= 2\rho^2\frac{\partial e^{-1/2\rho^2}}{\partial\rho^2}\rho^{2m}L + 2\rho^2 e^{-1/2\rho^2}\frac{\partial\rho^{2m}}{\partial\rho^2}L + 2\rho^2 e^{-1/2\rho^2}\rho^{2m}\frac{\partial L}{\partial\rho^2} \\ &\quad= -e^{-1/2\rho^2}\rho^{2m+2}L + 2me^{-1/2\rho^2}\rho^{2m}L - 2e^{-1/2\rho^2}\rho^{2m+2}L_{r-1}^{2m+1}(\rho^2).\end{aligned}$$

[2] §5.5.2

The first term here cancels with the ρ^2 term of a_+ in Equation 5.24, whereas the second term adds to the $2m$ term in Equation 5.24. Therefore, the full action of a_+ on $|\psi\rangle$ is:

$$\begin{aligned} a_+ |\psi\rangle &= 2me^{-1/2\rho^2}\rho^{2m-1}L_r^{2m}(\rho^2) - e^{-1/2\rho^2}\rho^{2m+1}L_{r-1}^{2m+1}(\rho^2) \\ &= e^{-1/2\rho^2}\rho^{2m-1}(2mL_r^{2m}(\rho^2) - \rho^2 L_{r-1}^{2m+1}(\rho^2)). \end{aligned}$$

Using once again the notation, $\alpha = 2m$, $x = \rho^2$ and $n = r$:

$$\begin{aligned} \alpha L_n^\alpha(x) - xL_{n-1}^{\alpha+1}(x) &= \alpha L_n^\alpha(x) - x(L_n^{\alpha+1}(x) - L_n^\alpha(x)) \\ &= (\alpha + x)L_n^\alpha(x) - (\alpha + n)L_{n-1}^\alpha(x) + (n - x)L_n^\alpha(x) \\ &= (\alpha + n)L_n^\alpha(x) - (\alpha + n)L_{n-1}^\alpha(x) \\ &= (\alpha + n)(L_n^\alpha(x) - L_{n-1}^\alpha(x)) \\ &= (\alpha + n)L_n^{\alpha-1}(x) \end{aligned}$$

where the followings identities for associated Laguerre polynomials have been used:

$$L_n^\alpha = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x)$$

and

$$xL_n^{\alpha+1}(x) = (\alpha + n)L_{n-1}^\alpha(x) - (n - x)L_n^\alpha(x).$$

Therefore:

$$a_+ |\psi\rangle = (2m + r)e^{-1/2\rho^2}\rho^{2m-1}L_r^{2m-1}(\rho^2),$$

and so the action of a_+ on the wavefunction in Equation 5.22 is given by:

$$\begin{aligned} \langle\rho, \theta| a_+ |r + 2m, r\rangle &= \frac{\Phi_r^{2m}}{\pi^{1/2}}(2m + r)\left(\frac{\Gamma(r + 1)}{\Gamma(2m + r + 1)}\right)^{1/2} e^{-1/2\rho^2}\rho^{2m-1}L_r^{2m-1}(\rho^2) \\ &= \frac{\Phi_r^{2m}}{\pi^{1/2}}(2m + r)^{1/2}\left(\frac{\Gamma(r + 1)}{\Gamma(2m + r)}\right)^{1/2} e^{-1/2\rho^2}\rho^{2m-1}L_r^{2m-1}(\rho^2) \\ &= \frac{\Phi_r^{2m}}{\Phi_r^{2m-1}}\sqrt{2m + r}\,\langle\rho, \theta|r + 2m - 1, r\rangle\,. \end{aligned}$$

The calculations for the operators a_-, $a_+^\dagger$ and $a_-^\dagger$ are all very similar (in fact, the calculations for a_- and $a_+^\dagger$ are much simpler, whereas the calculation for $a_-^\dagger$ proceeds in much the same way as for a_+). These calculations result in:

$$\begin{aligned} \langle\rho, \theta| a_- |r + 2m, r\rangle &= -\frac{\Phi_r^{2m}}{\Phi_{r-1}^{2m+1}}\sqrt{r}\,\langle\rho, \theta|r + 2m, r - 1\rangle \\ \langle\rho, \theta| a_+^\dagger |r + 2m, r\rangle &= \frac{\Phi_r^{2m}}{\Phi_r^{2m+1}}\sqrt{r + 2m + 1}\,\langle\rho, \theta|r + 2m + 1, r\rangle \\ \langle\rho, \theta| a_-^\dagger |r + 2m, r\rangle &= -\frac{\Phi_r^{2m}}{\Phi_{r+1}^{2m-1}}\sqrt{r + 1}\,\langle\rho, \theta|r + 2m, r + 1\rangle\,. \end{aligned}$$

These relations suggest an appropriate choice for the phase Φ_r^{2m} is $(-1)^r$. With the phase chosen in this way, the relations become:

$$\begin{aligned}
\langle\rho,\theta|\, a_+ \,|r+2m,r\rangle &= \sqrt{r+2m}\,\langle\rho,\theta|r+2m-1,r\rangle \\
\langle\rho,\theta|\, a_- \,|r+2m,r\rangle &= \sqrt{r}\,\langle\rho,\theta|r+2m,r-1\rangle \\
\langle\rho,\theta|\, a_+^\dagger \,|r+2m,r\rangle &= \sqrt{r+2m+1}\,\langle\rho,\theta|r+2m+1,r\rangle \\
\langle\rho,\theta|\, a_-^\dagger \,|r+2m,r\rangle &= \sqrt{r+1}\,\langle\rho,\theta|r+2m,r+1\rangle\,,
\end{aligned}$$

as required.

5.3.2 Momentum Wavefunction in Position Representation

From the $m>0$ case of Equation 5.13:

$$|p^2,m\rangle = \sum_r \lambda_r^+ \,|r+2m,r\rangle$$

where

$$\begin{aligned}
\lambda_r^+ &= N^+(p^2,m)p^{4m}\left(\frac{\Gamma(r+1)}{\Gamma(r+2m+1)}\right)^{1/2} L_r^{2m}(p^2) \\
&= \mathcal{N}^+(p^2,m)\sqrt{\pi}e^{-1/2p^2}p^{2m}\left(\frac{\Gamma(r+1)}{\Gamma(r+2m+1)}\right)^{1/2} L_r^{2m}(p^2).
\end{aligned}$$

from Equation 5.20. In the second line, the normalisation factor $N^+(p^2,m)$ has been replaced by $\mathcal{N}^+(p^2,m) := N^2(p^2,m)e^{1/2p^2}p^{2m}/\sqrt{\pi}$ in order that the terms involving p match those involving ρ ($N^+(p^2,m)$ is indeed proportional to p^{-2m} as was determined in Equation 5.21). And so:

$$\begin{aligned}
\langle\rho,\theta|p^2,m\rangle &= \sum_r \lambda_r^+ \,\langle\rho,\theta|r+2m,r\rangle \\
&= \sum_r \lambda_r^+ \frac{(-1)^{2m}}{\sqrt{\pi}}\left(\frac{\Gamma(r+1)}{\Gamma(r+2m+1)}\right)^{1/2} e^{-1/2\rho^2}\rho^{2m}L_r^{2m}(\rho^2) \\
&= \mathcal{N}^+\sum_r (-1)^r \frac{\Gamma(r+1)}{\Gamma(r+2m+1)} e^{-1/2(\rho^2+p^2)}(p^2\rho^2)^m L_r^{2m}(p^2)L_r^{2m}(\rho^2).
\end{aligned}$$

This matches the form of functions $K(x,y,t)$ in Magnus[3] [68], where K is obtained as a generating function of two associated Laguerre polynomials:

$$K(x,y,t) = \sum_{n=0}^{\infty} \frac{t^n n! e^{1/2(x+y)}}{\Gamma(n+\alpha+1)}(xy)^{\alpha/2}L_n^\alpha(x)L_n^\alpha(y),$$

[3]Page 242.

where $x = p^2$, $y = \rho^2$ $t = -1$, $n = r$ and $\alpha = 2m$. Therefore:

$$\langle \rho, \theta | p^2, m \rangle = \mathcal{N}^+(p^2, m) K(p^2, \rho^2, -1). \tag{5.25}$$

The function $K(x, y, t)$ is in fact a function involving modified Bessel functions, and is defined by Magnus [68] as:

$$K(x, y, t) = \frac{t^{-\alpha/2}}{1-t} \exp\left[-\frac{1}{2}(x+y)\frac{1+t}{1-t}\right] I_\alpha\left(2\frac{\sqrt{xyt}}{1-t}\right)$$
$$\Rightarrow K(p^2, \rho^2, -1) = \frac{(-1)^{-m}}{2} I_{2m}(ip\rho).$$

The modified Bessel function, since its argument is purely imaginary (for real p, ρ), can clearly be written instead as a Bessel function of the first kind $J_\alpha(x)$. Since $I_\alpha(x) = i^{-\alpha} J_\alpha(ix)$ by definition, K can instead be written as:

$$K(p^2, \rho^2, -1) = \frac{(-1)^{-m}}{2} i^{-2m} J_{2m}(p\rho).$$

The argument of the Bessel function J_α can be taken to be positive here since Bessel functions are even functions for even-integer values of α, which is equal to $2m$ with m integral[4]. Since $i^{-2m} = (-1)^m$ and $((-1)^m)^2 = 1$, K simplifies further to:

$$K(p^2, \rho^2, -1) = \tfrac{1}{2} J_{2m}(p\rho)$$

and so

$$\langle \rho, \theta | p^2, m \rangle = \frac{\mathcal{N}^+(p^2, m)}{2} J_{2m}(p\rho). \tag{5.26}$$

5.3.3 Check of $P_\pm$ Matrix Elements

For the purposes of checking the matrix elements $P_\pm$ in the position representation, the overall normalisation is not important. However, as an ansatz, a dependence on $e^{2mi\theta}$ is included. Therefore, in polar coordinates the states can be written as:

$$\langle \rho, \theta | p^2, m \rangle = J_{2m}(p\rho) e^{2mi\theta}.$$

The standard closure relation for Bessel functions is given by:

$$\int_0^\infty x J_\alpha(ux) J_\alpha(vx) dx = \frac{1}{u}\delta(u - v),$$

[4]Written as a Taylor series expansion, $J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{2n+\alpha}$ which is even for $\alpha = 2m$. This is confirmed by the Bessel "circuit relation" [68], wherein $J_\alpha(e^{n\pi i}x) = e^{n\alpha\pi i} J_\alpha(x)$. Replacing n with 1 and α with $2m$, this yields $J_{2m}(e^{\pi i}x) = e^{2m\pi i} J_{2m}(x)$ or $J_{2m}(-x) = J_{2m}(x)$ for integral m.

and so the completeness relation is given by::

$$\begin{aligned}\langle q^2, m|p^2, m\rangle &= \int_0^\infty \rho J_{2m}(q\rho)J_{2m}(p\rho)d\rho \\ &= \frac{1}{q}\delta(q-p).\end{aligned}$$

The matrix elements $P_\pm$ act as:

$$\begin{aligned}\langle\rho,\theta|\, P_\pm\, |p^2, m\rangle &= p\,\langle\rho,\theta|p^2, m\pm\tfrac{1}{2}\rangle \\ &= pJ_{2m\pm1}(p\rho)e^{i(2m\pm1)\theta}.\end{aligned}$$

Therefore:

$$\langle q^2, m\pm\tfrac{1}{2}|\, P_\pm\, |p^2, m\rangle = p\,\langle q^2, m\pm\tfrac{1}{2}|p^2, m\pm\tfrac{1}{2}\rangle\,.$$

Following similar arguments to those that lead up to Equation 5.24, in polar coordinates the $P_\pm$ operators can be expressed as:

$$P_\pm \to \tfrac{i}{\rho}(\rho\partial_\rho \mp 2m)$$

where it has been assumed that ∂_θ acting on an angular wavefunction of the form $e^{2im\theta}$ gives $2im$. Therefore:

$$\begin{aligned}\langle q^2, m\pm\tfrac{1}{2}|\, P_\pm\, |p^2, m\rangle &= \int \rho d\rho\,\langle q^2, m\pm\tfrac{1}{2}|\rho\theta\rangle\,\langle\rho,\theta|\, P_\pm\, |p^2, m\rangle \\ &= i\int d\rho J_{2m\pm1}(q\rho)(\rho\partial_\rho \mp 2m)J_{2m}(p\rho) \\ &= i\int \rho d\rho J_{2m\pm1}(q\rho)(\mp pJ_{2m\pm1}(p\rho)).\end{aligned}$$

The last line is obtained by the fact that Bessel functions obey recurrence relations for derivatives of the form:

$$\frac{d}{dx}J_n(\alpha x) = -\alpha J_{n+1}(\alpha x) + \frac{n}{x}J_n(\alpha x) \tag{5.27a}$$

$$\text{and}\quad \frac{d}{dx}J_n(\alpha x) = \alpha J_{n-1}(\alpha x) - \frac{n}{x}J_n(\alpha x). \tag{5.27b}$$

Substituting back the appropriate labels gives:

$$\rho\partial_\rho J_{2m}(p\rho) \mp 2mJ_{2m}(p\rho) = \mp p\rho J_{2m\pm1}(p\rho).$$

Finally then:

$$\langle q^2, m\pm\tfrac{1}{2}|\, P_\pm\, |p^2, m\rangle = \mp i\frac{p}{q}\delta(q-p). \tag{5.28}$$

This confirms the dependence of $\langle\rho,\theta,p^2,m\rangle$ on a factor of $e^{2mi\theta}$. This is not surprising, since the Jacobi-Anger identity suggests such a phase in the conversion from Cartesian to polar coordinates. The Jacobi-Anger identity is given by:

$$e^{ip\rho\cos\theta} = \sum_{2m=-\infty}^{+\infty} i^{2m}J_{2m}(p\rho)e^{2mi\theta}.$$

5.3.4 Evaluation of $X_\pm$ Matrix Elements

In polar coordinates, the operators $X_\pm$ are given by:

$$X_\pm = \rho e^{\pm i\theta}.$$

Therefore the matrix elements are given by:

$$\begin{aligned}\langle q^2, m \pm \tfrac{1}{2}| \, X_\pm \, |p^2, m\rangle &= \int \rho d\rho \, \langle q^2, m \pm \tfrac{1}{2}|\rho, \theta\rangle \, \langle \rho, \theta| \, X_\pm \, |p^2, m\rangle \\ &= \int \rho d\rho (\rho J_{2m\pm1}(q\rho)) J_{2m}(p\rho),\end{aligned}$$

where the azimuthal quantum number label is taken to have been dealt with by the completeness relation (an assumption that will continue throughout the rest of this chapter). Recalling the Bessel function recurrence relations for derivatives in Equations 5.27, $J_{2m\pm1}$ can be written as:

$$\rho J_{2m\pm1}(q\rho) = \frac{2m}{q} J_{2m}(q\rho) \mp \partial_q J_{2m}(q\rho).$$

And so, the calculations for the matrix elements become:

$$\begin{aligned}\langle q^2, m \pm \tfrac{1}{2}| \, X_\pm \, |p^2, m\rangle &= \int \rho d\rho \left(\frac{2m}{q} J_{2m}(q\rho) \mp \partial_q J_{2m}(q\rho) \right) J_{2m}(p\rho) \\ &= \frac{2m}{q^2}\delta(q-p) \mp \partial_q \frac{1}{q}\delta(q-p) \\ &= \frac{2m \pm 1}{q^2}\delta(q-p) \mp \frac{1}{q}\partial_q \delta(q-p) \\ &\equiv \frac{2m}{q^2}\delta(q-p) \pm \frac{1}{p}\partial_p \delta(q-p).\end{aligned}$$

The last line has been determined from the fundamental δ function lemma[5]:

$$f(x)\partial_x \delta(x-y) = f(y)\partial_x \delta(x-y) - (\partial_y f(y)\delta(x-y)) \tag{5.29a}$$

$$= -f(y)\partial_y \delta(x-y) - (\partial_y f(y))\delta(x-y). \tag{5.29b}$$

5.3.5 Check of Heisenberg Algebra $[X_+, P_-] \propto 2iI$

The Heisenberg algebra can now be checked in the momentum basis, for operators X_+ and P_-. The commutation relation is given by:

$$[X_+, P_-] = X_+P_- - P_-X_+.$$

[5]Proved by integrating by parts. See, for example, [45].

The terms will be calculated individually. The first term is given by:

$$\begin{aligned}
\langle q^2, m|X_+P_- \left|p^2, m\right\rangle &= \int udu \left\langle q^2, m\right| X_+ \left|u^2, m-\tfrac{1}{2}\right\rangle \left\langle u^2, m-\tfrac{1}{2}\right| P_- \left|p^2, m\right\rangle \\
&= \int udu \left(\frac{2m}{q^2}\delta(q-u) - \frac{1}{q}\partial_q\delta(q-u)\right) i\frac{p}{u}\delta(u-p) \\
&= i\frac{2mp}{q^2}\int udu\delta(q-u)\delta(u-p) - i\frac{p}{q}\partial_q\int du\delta(q-u)\delta(u-p) \\
&= i\frac{2mp}{q^2}\delta(q-p) - i\frac{p}{q}\partial_q\delta(q-p) \\
&= i\frac{2m}{q}\delta(q-p) - i\frac{q}{p}\partial_q\delta(q-p) \\
&= i\frac{2m}{q}\delta(q-p) - i\frac{1}{p}\delta(q-p) + i\partial_p\delta(q-p),
\end{aligned}$$

where the δ function lemma of Equations 5.29 has been used. In the calculation of the second term, the alternative form of X_+ will be used. The second term is then given by:

$$\begin{aligned}
\left\langle q^2, m\right| P_-X_+ \left|p^2, m\right\rangle &= \int udu \left\langle q^2, m\right| P_- \left|u^2, m+\tfrac{1}{2}\right\rangle \left\langle u^2, m+\tfrac{1}{2}\right| X_+ \left|p^2, m\right\rangle \\
&= i\int udu\frac{u}{q}\delta(q-u)\left(\frac{2m}{u^2}\delta(u-p) + \frac{1}{p}\partial_p\delta(u-p)\right) \\
&= i\frac{2m}{q}\delta(q-p) + i\frac{1}{qp}\partial_p\int u^2du\delta(q-u)\delta(u-p) \\
&= i\frac{2m}{q}\delta(q-p) + i\frac{q}{p}\partial_p\delta(q-p) \\
&= i\frac{2m}{q}\delta(q-p) + i\frac{1}{q}\delta(q-p) + i\partial_p\delta(q-p),
\end{aligned}$$

where, again, in the last line the δ lemma of Equations 5.29 has been used, as well as the fact that $\partial_q\delta(q-p) = -\partial_p\delta(q-p)$. The commutation relation therefore becomes:

$$\begin{aligned}
\left\langle q^2, m\right| \left[X_+, P_-\right] \left|p^2, m\right\rangle &= -i\frac{2}{q}\delta(q-p) \\
&= -2i \left\langle q^2, m\right| I \left|p^2, m\right\rangle .
\end{aligned}$$

This calculation indeed shows that the Heisenberg algebra is consistent in this framework (up to sign).

5.3.6 Evaluation of $X_\pm^2$ and X_+X_-

The operators $X_\pm^2$ and X_+X_- can now be evaluated. For $X_\pm^2$:

$$\begin{aligned}
\left\langle q^2, m\pm 1\right| X_\pm^2 \left|p^2, m\right\rangle &= \int \rho d\rho \left\langle q^2, m\pm 1|\rho\theta\right\rangle \left\langle \rho, \theta\right| X_\pm^2 \left|p^2, m\right\rangle \\
&= \int \rho d\rho(\rho^2 J_{2m\pm 2}(q\rho))J_{2m}p\rho.
\end{aligned}$$

The Bessel function recurrence relation of Equation 5.27a can be adapted to this situation as follows:

$$\begin{aligned}\alpha^2 J_{p+2}(\alpha x) &= \frac{p+1}{x}\alpha J_{p+1}(\alpha x) - \partial_x \alpha J_{p+1}(\alpha x) \\ &= \frac{(p+1)p}{x^2} J_p(\alpha x) - \frac{p+1}{x}\partial_x J_p(\alpha x) - \partial_x(\frac{p}{x} J_p(\alpha x) - \partial_x J_p(\alpha x)) \\ &= \frac{p^2+2p}{x^2} J_p(\alpha x) - \frac{2p+1}{x}\partial_x J_p(\alpha x) + \partial_{xx} J_p(\alpha x).\end{aligned}$$

Similarly, Equation 5.27b can be adapted as follows:

$$\begin{aligned}\alpha^2 J_{p-2}(\alpha x) &= \partial_x \alpha J_{p-1}(\alpha x) + \frac{(p-1)}{x}\alpha J_{p-1}(\alpha x) \\ &= \partial_x(\partial_x J_p(\alpha x) + \frac{p}{x} J_p(\alpha x)) + \frac{p-1}{x}\partial_x J_p(\alpha x) + \frac{(p-1)p}{x^2} J_p(\alpha x)) \\ &= \frac{p^2-2p}{x^2} J_p(\alpha x) + \frac{2p-1}{x}\partial_x J_p(\alpha x) + \partial_{xx} J_p(\alpha x).\end{aligned}$$

And so, for X_+^2:

$$\begin{aligned}&\langle q^2, m+1 | X_+^2 \, | p^2, m\rangle = \\ &\qquad = \int \rho d\rho \left(\frac{4m^2+4m}{q^2} J_{2m}(q\rho) - \frac{4m+1}{q} J_{2m}(q\rho) + \partial_{qq} J_{2m}(q\rho) \right) \\ &\qquad = \frac{4m^2+4m}{q^3}\delta(q-p) - \frac{4m+1}{q}\partial_q \frac{1}{q}\delta(q-p) + \partial_{qq}\frac{1}{q}\delta(q-p) \\ &\qquad = \frac{4m^2+8m+1}{q^3}\delta(q-p) - \frac{4m+3}{q^2}\partial_q\delta(q-p) + \frac{1}{q}\partial_{qq}\delta(q-p),\end{aligned}$$

where in the last line, the fact that:

$$\begin{aligned}\partial_{qq} q^{-1}\delta(q-p) &= \partial_q(-q^{-2}\delta(q-p) + q^{-1}\delta(q-p)) \\ &= 2q^{-3}\delta(q-p) - 2q^{-2}\partial_q\delta(q-p) + q^{-1}\partial_{qq}\delta(q-p)\end{aligned}$$

has been used. A similar calculation to the one above for X_+^2 shows that for X_-^2:

$$\begin{aligned}&\langle q^2, m-1 | X_-^2 \, | p^2, m\rangle = \\ &\qquad = \frac{4m^2-8m-1}{q^3}\delta(q-p) + \frac{4m-3}{q^2}\partial_q\delta(q-p) + \frac{1}{q}\partial_{qq}\delta(q-p).\end{aligned}$$

And finally, a similar calculation shows that X_+X_- is given by:

$$\begin{aligned}&\langle q^2, m-1 | X_+X_- \, | p^2, m\rangle = \\ &\qquad = \frac{4m^2}{q^3}\delta(q-p) - \frac{2m}{qp}\partial_p\delta(q-p) - \frac{2m}{q^2}\partial_q\delta(q-p) + \frac{1}{p}\partial_q\partial_p\delta(q-p).\end{aligned} \tag{5.30}$$

5.3.7 Evaluation of $Z'_{\pm}$ and U

From Equation 5.28, the operators $P_{\pm}$ both yield $-\frac{p^2}{q}\delta(q-p)$. Therefore, the Z-type Heisenberg generators defined in Equations 5.12 can now be evaluated. These are given by:

$$\begin{aligned} Z'_{+} &= \tfrac{1}{2}(X_{+}^2 + P_{+}^2) \\ &= \frac{4m^2+8m+1}{2q^3}\delta(q-p) - \frac{4m+3}{2q^2}\partial_q\delta(q-p) + \frac{1}{2q}\partial_{qq}\delta(q-p) - \frac{p^2}{2q}\delta(q-p). \end{aligned}$$

Similarly:

$$\begin{aligned} Z'_{-} &= \tfrac{1}{2}(X_{-}^2 + P_{-}^2) \\ &= \frac{4m^2-8m-1}{2q^3}\delta(q-p) + \frac{4m-3}{2q^2}\partial_q\delta(q-p) + \frac{1}{2q}\partial_{qq}\delta(q-p) - \frac{p^2}{2q}\delta(q-p). \end{aligned}$$

From Equation 5.11, $U = 2\epsilon + \frac{1}{2}(X_{+}X_{-} + P_{+}P_{-})$. The operator $X_{+}X_{-}$ is given in Equation 5.30. From 5.28, the operator $P_{+}P_{-}$ is clearly given by $\frac{p^2}{q}\delta(q-p)$. Therefore, the operator U is given by:

$$\begin{aligned} &U = \\ &\frac{4\epsilon+p^2}{2q}\delta(q-p) + \frac{2m^2}{q^3}\delta(q-p) - \frac{m}{qp}\partial_p\delta(q-p) - \frac{m}{q^2}\partial_q\delta(q-p) + \frac{1}{2p}\partial_q\partial_p\delta(q-p). \end{aligned}$$

Chapter 6

Granularity of Space-Time

This chapter considers the Schrödinger-Robertson inequality as it applies to Born reciprocity. In Section 6.1 the Schrödinger-Robertson Inequality is introduced and then defined for the case of 3-phase space (cf. Section 1.3). The determinant of the covariance matrix Σ is quaplectic invariant and this is proven in Section 6.1.1. In Section 6.2 the square-root of the inequality is taken for states that are uncorrelated, resulting in a natural extension of the inequality of Heisenberg's uncertainty principle. Furthermore, the implications of the quaplectic invariance of $\det(\Sigma)$ is explored for general states and in particular for Schrödinger-Robertson minimal uncertainty states, which are states which saturate the inequality.

The quaplectic Lie algebra is given in Section 3.1.3.1. The particular generators of importance in this chapter are the Z^{μ}, $\overline{Z}^{\mu}$ of Equation 3.18, the $Z^{\mu}{}_{\nu}$ of Equation 3.23, the $\mathcal{U}(1,3)$ generators $E^{\mu}{}_{\nu}$, and the auxiliary generators $\varepsilon^{\mu}{}_{\nu}$ defined in terms of $E^{\mu}{}_{\nu}$ and $Z^{\mu}{}_{\nu}$ in Equation 3.24[1]. As in Equation 3.27, the $E^{\mu}{}_{\nu}$ can be given in terms of the $Z^{\mu}{}_{\nu}$ and $\varepsilon^{\mu}{}_{\nu}$ as:

$$E^{\mu}{}_{\nu} = \varepsilon^{\mu}{}_{\nu} + Z^{\mu}{}_{\nu}.$$

Due to the construction of the representations of the quaplectic group along the lines of the Mackey theory, the quaplectic generators are given by $E^{\mu}{}_{\nu}$. Henceforth for this chapter, only the scalar case will be considered such that $\varepsilon^{\mu}{}_{\nu} = 0$. Therefore, the quaplectic generators are given by:

$$E^{\mu}{}_{\nu} = \tfrac{1}{2}\left\{\overline{Z}^{\mu}, Z_{\nu}\right\}.$$

Appropriate representations of the Heisenberg group are associated with the "relativistic oscillator". The Heisenberg generators are now associated with Fock generators a_{μ} and $a_{\mu}^{\dagger}$, such that Z_0 is associated with a creation operator $a_0^{\dagger}$ whereas the generators Z_i are associated with annihilation operators a_i. Conversely, $\overline{Z}_0$ is associated with an annihilation operator a_0 whereas the generators $\overline{Z}_i$ are associated with creation operators $a_i^{\dagger}$. These choices are reflective of the sign change in the definition of Z_{μ} and $\overline{Z}_{\mu}$ in terms of position and momentum generators X_{μ} and

[1]The cross-referenced equation actually defines $W^{\mu}{}_{\nu}$, and the $\varepsilon^{\mu}{}_{\nu}$ are the representations of those generators.

P_μ between the time component and the space components. The zero occupancy state is defined in the number basis as:

$$\begin{aligned}\Psi_{n_0=n_1=n_2=n_3=0}(x^\mu) &\propto \exp\left(-\tfrac{1}{2}(x^0)^2\right)\cdot\exp\left(\tfrac{1}{2}x^i x^j\eta_{ij}\right)\\ &= \exp\left(-\tfrac{1}{2}((x^0)^2+(x^1)^2+(x^2)^2+(x^3)^2)\right).\end{aligned}$$

The Hilbert space is that of $\mathcal{H}(4)$ (cf. Section 2.6.2.2); namely square-integrable functions on space-time.

6.1 Schrödinger-Robertson Inequality

The Schrödinger-Robertson Inequality [71] is given by:

$$\det(\Sigma) \geq \det(C), \tag{6.1}$$

for $2n$ observables $A_1, A_2, ..., A_{2n}$, where

$$\begin{aligned}\Sigma_{ij} &= \mathrm{cov}\left(A_i, A_j\right) \text{ is the } \textit{covariance} \text{ matrix}\\ C_{ij} &= -\frac{i}{2}\left\langle[A_i, A_j]\right\rangle\\ \text{and } \mathrm{cov}\left(A,B\right) &= \frac{1}{2}\left(\langle AB\rangle + \langle BA\rangle\right) - \langle A\rangle\langle B\rangle,\end{aligned}$$

for observables A, B. The covariance matrix is sometimes called the "uncertainty matrix". The expression for the Schrödinger-Robertson Inequality can be reduced to the particular case of $n = 1$; giving the Schrödinger Inequality of [73]. This is given by:

$$\Delta^2 A\cdot\Delta^2 B \geq \frac{1}{4}\left|\langle[A,B]\rangle\right|^2 + \mathrm{cov}\left(A,B\right), \tag{6.2}$$

noting that $\Delta^2 A = \mathrm{cov}(A, A)$. The Inequality 6.1 is sometimes simply called the "Robertson inequality".

For eight dimensional space-time-momentum-energy with observables X^0, X^1, X^2, X^3 and P^0, P^1, P^2, P^3, Σ can be written as:

$$\Sigma = \begin{pmatrix} \Sigma^{\mu\nu} & \Sigma^{\mu\nu'} \\ \Sigma^{\mu'\nu} & \Sigma^{\mu'\nu'} \end{pmatrix}$$

where $\mu' = \mu+4 = 4, 5, 6, 7$ and similarly for ν'. Where indicated, the dashed index will refer to energy-momentum dimensions whereas the undashed index could refer to either. Space-time dimensions will always be undashed. Then C can be written as:

$$\begin{aligned}C &= \begin{pmatrix} C^{\mu\nu} & C^{\mu\nu'} \\ C^{\mu'\nu} & C^{\mu'\nu'} \end{pmatrix}\\ &= -\frac{i}{2}\begin{pmatrix} <[X^\mu, X^\nu]> & <[X^\mu, P^\nu]> \\ <[P^\mu, X^\nu]> & <[P^\mu, P^\nu]> \end{pmatrix}\\ &= \frac{1}{2}\hbar\begin{pmatrix} 0 & \eta^{\mu\nu} \\ -\eta^{\mu\nu} & 0 \end{pmatrix},\\ &= \tfrac{1}{2}\hbar J,\end{aligned}$$

where

$$J := \begin{pmatrix} 0 & \eta_{4\times 4} \\ -\eta_{4\times 4} & 0 \end{pmatrix} \tag{6.3}$$

is the symplectic metric[2]. Therefore:

$$\det(C) = (\tfrac{1}{2}\hbar)^8,$$

and the Schrödinger-Robertson Inequality for any state in 3-phase space is given by:

$$\det(\Sigma) \geq (\tfrac{1}{2}\hbar)^8. \tag{6.4}$$

6.1.1 Quaplectic Invariance of $\det(\Sigma)$

Homogeneous quaplectic transformations $U \in \mathcal{U}(1,3)$ can be written as:

$$\begin{aligned} U &= e^{(\omega^{\mu}{}_{\nu} E^{\nu}{}_{\mu})} \\ &\simeq I + \omega^{\mu}{}_{\nu} E^{\nu}{}_{\nu}, \end{aligned}$$

where $E^{\mu}{}_{\nu}$ are the generators of the Lie algebra of $\mathcal{U}(1,3)$ (see Section 3.1.3.1). Similarly:

$$\begin{aligned} U^{\dagger} &= U^{-1} \\ &= e^{-(\omega^{\mu}{}_{\nu} E^{\nu}{}_{\mu})} \\ &\simeq I - \omega^{\mu}{}_{\nu} E^{\nu}{}_{\nu}. \end{aligned}$$

Define $\mathcal{E} := \omega^{\mu}{}_{\nu} E^{\nu}{}_{\mu}$. Then:

$$U^{-1} A U \simeq A - [\mathcal{E}, A]\,.$$

Similarly for $\det(\Sigma)$:

$$\begin{aligned} \det(\Sigma) &\to \det(U^{-1}\Sigma U) \\ &\simeq \det(\Sigma - [\mathcal{E}, \Sigma]). \end{aligned}$$

The following definition is useful:

$$[\mathcal{E}, \mathrm{cov}(A, B))] = \mathrm{cov}([\mathcal{E}, A]\,, B) + \mathrm{cov}(A, [\mathcal{E}, B]). \tag{6.5}$$

Proof:

$$\begin{aligned} \text{LHS} =& \tfrac{1}{2}\left[\mathcal{E}, \langle AB\rangle\right] + \tfrac{1}{2}\left[\mathcal{E}, \langle BA\rangle\right] - \left[\mathcal{E}, \langle A\rangle \langle B\rangle\right]; \\ \text{RHS} =& \tfrac{1}{2}\left\langle [\mathcal{E}, A]\, B\right\rangle + \tfrac{1}{2}\left\langle B\,[\mathcal{E}, A]\right\rangle - \left\langle [\mathcal{E}, A]\right\rangle \langle B\rangle + \tfrac{1}{2}\left\langle A\,[\mathcal{E}, B]\right\rangle + \\ & + \tfrac{1}{2}\left\langle [\mathcal{E}, B]\, A\right\rangle - \langle A\rangle \left\langle [\mathcal{E}, B]\right\rangle \\ =& \tfrac{1}{2}\langle \mathcal{E} AB\rangle - \tfrac{1}{2}\langle BA\mathcal{E}\rangle - \tfrac{1}{2}\langle AB\mathcal{E}\rangle + \tfrac{1}{2}\langle \mathcal{E} BA\rangle - \langle \mathcal{E} A\rangle \langle B\rangle + \langle A\rangle \langle B\mathcal{E}\rangle \\ =& \tfrac{1}{2}(\mathcal{E}\langle AB\rangle - \langle BA\rangle \mathcal{E}) + \tfrac{1}{2}(\mathcal{E}\langle BA\rangle - \langle AB\rangle \mathcal{E}) - \mathcal{E}\langle A\rangle \langle B\rangle + \langle A\rangle \langle B\rangle \mathcal{E} \\ =& \text{LHS}. \end{aligned}$$

[2]This differs to the symplectic metric ω in Equation 1.4 by elementary row and column permutations only.

Furthermore, from Equations 3.19 and 3.21:

$$\begin{aligned}
[\mathcal{E}, X^\mu] &= \omega^\rho{}_\sigma \left[E^\sigma{}_\rho, X^\mu\right] \\
&= \frac{1}{\sqrt{2}}(\omega^\rho{}_\sigma \left[E^\sigma{}_\rho, Z^\mu\right] + \omega^\rho{}_\sigma \left[E^\sigma{}_\rho, \overline{Z}^\mu\right]) \\
&= \frac{1}{\sqrt{2}}\omega^\rho{}_\sigma(-\eta^{\sigma\mu} Z_\rho + \delta_\rho{}^\mu \overline{Z}^\sigma) \\
&= \frac{1}{\sqrt{2}}\omega^\rho{}_\sigma(-\eta^{\sigma\mu}\eta_{\rho\nu} Z^\nu + \delta_\rho{}^\mu \overline{Z}^\sigma) \\
&= i\omega^\mu{}_\sigma P^\sigma.
\end{aligned}$$

A similar calculation shows that:

$$[\mathcal{E}, P^\mu] = -i\omega^\mu{}_\sigma X_\sigma.$$

Then (with $\omega \equiv i\omega^\mu{}_\sigma$ and $\tilde{\omega} \equiv i\omega^\nu{}_\sigma$ for brevity of notation):

$$\begin{aligned}
[\mathcal{E}, \Sigma] &= \begin{pmatrix} [\mathcal{E}, \Sigma^{\mu\nu}] & [\mathcal{E}, \Sigma^{\mu\nu'}] \\ [\mathcal{E}, \Sigma^{\mu'\nu}] & [\mathcal{E}, \Sigma^{\mu'\nu'}] \end{pmatrix} \\
&= \begin{pmatrix} \omega\mathrm{cov}(P^\sigma, X^\nu) + \tilde{\omega}\mathrm{cov}(X^\mu, P^\sigma) & \omega\mathrm{cov}(P^\sigma, P^\nu) - \tilde{\omega}\mathrm{cov}(X^\mu, X^\sigma) \\ -\omega\mathrm{cov}(X^\sigma, X^\nu) + \tilde{\omega}\mathrm{cov}(P^\mu, P^\sigma) & -\omega\mathrm{cov}(X^\sigma, P^\nu) - \tilde{\omega}\mathrm{cov}(P^\mu, X^\sigma) \end{pmatrix} \\
&= \begin{pmatrix} 0 & i\omega^\mu{}_{\sigma'} \\ -i\omega^\mu{}_\sigma & 0 \end{pmatrix} \begin{pmatrix} \Sigma^{\sigma\nu} & \Sigma^{\sigma\nu'} \\ \Sigma^{\sigma'\nu} & \Sigma^{\sigma'\nu'} \end{pmatrix} + \begin{pmatrix} \Sigma^{\mu\sigma} & \Sigma^{\mu\sigma'} \\ \Sigma^{\mu'\sigma} & \Sigma^{\mu'\sigma'} \end{pmatrix} \begin{pmatrix} 0 & -i\omega^\nu{}_\sigma \\ i\omega^\nu{}_{\sigma'} & 0 \end{pmatrix}
\end{aligned}$$

By denoting the first traceless matrix as Ω and the second as $\widetilde{\Omega}$, this becomes:

$$\Sigma - [\mathcal{E}, \Sigma] = \Sigma + \Omega \cdot \Sigma + \Sigma \cdot \widetilde{\Omega}$$

The determinant of $\Sigma - [\mathcal{E}, \Sigma]$ is therefore given by:

$$\begin{aligned}
\det\left(\Sigma - [\mathcal{E}, \Sigma]\right) &= \det\left(\Sigma\right) \cdot \det\left(1 + \Sigma^{-1}\left[\mathcal{E}, \Sigma\right]\right) \\
&= \det\left(\Sigma\right) \cdot \det\left(1 + \Sigma^{-1}\Omega\Sigma + \widetilde{\Omega}\right) \\
&= \det\left(\Sigma\right) \exp \mathrm{tr} \log(1 + \Sigma^{-1}\Omega\Sigma + \widetilde{\Omega}) \\
&\simeq \det(\Sigma) \cdot \exp \mathrm{tr}(\Sigma^{-1}\Omega\Sigma + \widetilde{\Omega}) \\
&= \det(\Sigma) \cdot \exp \mathrm{tr}(\Omega + \widetilde{\Omega}) \\
&= \det(\Sigma).
\end{aligned}$$

Thus it can be stated that $\det(\Sigma)$ is invariant under transformations $U \in \mathcal{U}(1,3)$.

A similar calculation shows that $\det(\Sigma)$ is invariant under transformations $h \in \mathcal{H}(4)$. These transformations are of the form:

$$\begin{aligned}
h &= e^{(\zeta^\mu Z_\mu + \overline{\zeta}^\mu \overline{Z}_\mu)} \\
&\simeq I + \zeta^\mu Z_\mu + \overline{\zeta}^\mu \overline{Z}_\mu.
\end{aligned}$$

All of the brackets in the right-hand-side of Equation 6.5 relevant to this calculation will be proportional to I due to their being of the form of the Heisenberg commutation relations. The covariance therefore vanishes, and so $\det(\Sigma)$ is invariant under Heisenberg transformations. Thus, $\det(\Sigma)$ is invariant under all quaplectic transformations.

6.2 Space-Time Granularity

For uncorrelated states, where $\mathrm{cov}(X^\mu, X^\nu)$ and $\mathrm{cov}(P^\mu, P^\nu)$ vanish and for $\mu \neq \nu$ $\mathrm{cov}(X^\mu, P^\nu)$ vanishes, the square root of Inequality 6.4 can be taken, giving:

$$\Delta X^0 \Delta X^1 \Delta X^2 \Delta X^3 \Delta P^0 \Delta P^1 \Delta P^2 \Delta P^3 \geq (\tfrac{1}{2}\hbar)^4;$$

a kind of "granularity" in space-time, involving a natural extension of the inequality of Heisenberg's uncertainty principle. Furthermore, the invariance of $\det(\Sigma)$ under quaplectic transformations Q indicates that there are many physically equivalent states $|\psi'\rangle = Q\,|\psi\rangle$ which are related by quaplectic transformations and have the same measure of spreading. Included in these states are correlated states as well as uncorrelated states, including in particular states which have a space-momentum correlation. In other words, for a given uncertainty measure σ there is no natural distinction between uncorrelated states and correlated states, and this is due to the symmetry of the quaplectic group. Additionally, a state squeezed in position may have the same status as a state squeezed in momentum since there is the possibility that there exists a quaplectic transformation which transforms one state to the other.

The covariance matrix Σ is positive semi-definite since all covariance matrices must be positive semi-definite (indeed, all positive semi-definite matrices are also covariance matrices). Additionally, $\det C > 0$ and so it has no eigenvalues equal to zero. Therefore, Σ is in fact positive definite. A well-known result of matrix algebra called "Williamson's theorem" (cf. §8.3 of [24] and §4 of [30]) states that any positive definite matrix such as Σ can be diagonalised by a symplectic operator $S \in \mathcal{S}p(2n)$ (where $n = 4$ in this case):

$$\Sigma' = S^T \Sigma S;$$
$$\text{where } S^T J S = S$$

and J is defined in Equation 6.3. The diagonalisation of Σ can equally well be carried out by an appropriate unitary state transformation U on a state $|\psi\rangle$ [75]. First, define new indices $K, L, M, N = 0, 1..., 7$ which is to run over both (space-time) $\kappa = 0, 1, 2, 3$ and (energy-momentum) $\kappa' = 4, 5, 6, 7$. Then Heisenberg operators Z_K are given by:

$$Z_K = \begin{pmatrix} Z_\kappa \\ Z_{\kappa'} \end{pmatrix} = \begin{pmatrix} Z_\kappa \\ \overline{Z}_\kappa \end{pmatrix}.$$

New generators Z_{KL} can then be defined in terms of these generators as follows (cf. Equation 3.23):

$$Z_{KL} := \tfrac{1}{2}\{Z_K, Z_L\}.$$

These generators satisfy the following commutation relations:

$$[Z_{KL}, Z_{MN}] = -(J_{LM} Z_{KN} + J_{KM} Z_{LN} + J_{LN} Z_{KM} + J_{KN} Z_{LM}) \tag{6.6a}$$
$$[Z_{KL}, Z_M] = -(J_{LM} Z_K + J_{KM} Z_L) \tag{6.6b}$$
$$[Z_K, Z_L] = -J_{KL} I, \tag{6.6c}$$

and they can be used to diagonalise ψ:

$$|\psi'\rangle = U\,|\psi\rangle\,, \qquad U = e^{\frac{1}{2}\phi^{KL}Z_{KL}}.$$

Now consider the Schrödinger-Robertson minimal uncertainty states. These are states for which the Schrödinger-Robertson inequality of 6.1 is saturated, so that $\det(\Sigma) = \det(C)$. Sometimes these are called "multimode squeezed states" [75]. These states are of the form:

$$|\phi, \zeta\rangle = e^{\frac{1}{2}\phi^{KL}Z_{KL}} \cdot e^{\zeta^K Z_K}\,|0\rangle\,, \tag{6.7}$$

for parameters ϕ^{KL} and ζ^K. The zero occupancy state was defined above to be:

$$\langle x^0, x^1, x^2, x^3|0\rangle = \Psi_{n_0=n_1=n_2=n_3=0}(x^0, x^2, x^2, x^3).$$

The quaplectic symmetry and the principle of reciprocity leads irrevocably to the contention that:

> Different, but unitarily equivalent, minimal uncertainty states $|\phi, \zeta\rangle$ and $|\phi', \zeta'\rangle = U\,|\phi, \zeta\rangle$ only correspond to physically distinct semiclassical limits if the unitary transformation U relating them does *not* belong to the quaplectic group. [42]

As such, in the case of a scalar quaplectic system, there should be various physically different classes of minimal uncertainty states, parametrised by the geometry of the homogeneous space:

$$\mathcal{S}p(8, \mathbb{R}) \ltimes \mathcal{H}(4)/\mathcal{U}(1,3) \ltimes \mathcal{H}(4) \cong \mathcal{S}p(8, \mathbb{R})/\mathcal{U}(1,3). \tag{6.8}$$

General oscillator coherent states are defined by their characteristic of diagonalising a linear combination of creation and annihilation operators:

$$\alpha a + \beta a^\dagger\,|z\rangle = z\,|z\rangle$$

for constants α, β. To the extent that the quantities are "sharp" subject to the Heisenberg uncertainty principle, position and momentum quantities can be associated via the expectation values $\langle X\rangle$ and $\langle P\rangle$. Then pure position and momentum states arise for inadmissible values of the ratio α/β [75]. But the quaplectic group can "rotate" unitarily between position and momentum states. Therefore, the attribution of these quantities is purely frame-dependent.

A set of orbits for the homogeneous space of Equation 6.8 is parametrised by the states:

$$|\Phi\rangle = e^{\frac{1}{2}\phi^{\mu\nu}Z_{\mu\nu}} \cdot e^{\frac{1}{2}\overline{\phi}^{\mu\nu}\overline{Z}_{\mu\nu}}\,|\Psi\rangle \tag{6.9}$$

for a cyclic vector $|\Psi\rangle$. The limit $\hbar \to 0$ takes one from the quantum to the classical regime and the limit $b \to \infty$ takes one from the reciprocally relativistic regime to the special relativistic regime. These two limits together comprise the appropriate "semiclassical" limit. In these limits (cf. Section 4.1.4), the quaplectic group contracts to the Poincaré group generated by $L_{\mu\nu}$ and P_μ (an equally valid

choice would be $L_{\mu\nu}$ and X_μ). However, this set of generators is augmented by the set of ten $M^\circ_{\mu\nu}$ which are the contracted $M_{\mu\nu}$ generators. Therefore states of a quaplectically covariant system should be characterised by some eigenvalues $m_{\mu\nu}$ of these generators in the limit $b \to \infty$. In the scalar case, the tensor is essentially given by $X_\mu X_\nu + P_\mu P_\nu$. The commutation relations (Equations 6.6) of the symplectic algebra are given by:

$$\left[\overline{Z}_{\mu\nu}, Z_{\rho\sigma}\right] = \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho},$$

where the fact that $M_{\mu\nu} := E_{\mu\nu} + E_{\nu\mu}$, from Section 3.1.3.1. In the limit $b \to \infty$ there are two possibilities. In the first possibility, the contracted $Z^\circ_{\mu\nu}$, $\overline{Z}^\circ_{\mu\nu}$ become abelian and the physical states would become eigenstates with some complex eigenvalues $z_{\mu\nu}, \overline{z}_{\mu\nu}$. In the second possibility the contracted generators fulfil a 10 dimensional Heisenberg algebra with central generator $U := E^\mu{}_\mu$:

$$\left[Z^\circ_{\mu\nu}, \overline{Z}^\circ_{\rho\sigma}\right] = (\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho})U.$$

In this possibility, the states in Equation 6.9 with $|\Psi\rangle = |0\rangle$ have the structure of standard coherent states with expectation values $\langle Z^\circ_{\mu\nu}\rangle$ and $\langle \overline{Z}^\circ_{\mu\nu}\rangle$. From Equations 6.6 both possibilities lead to the consideration of the sharp quantities:

$$R_{\mu\nu} := \langle X_\mu P_\nu + X_\nu P_\mu \rangle ,$$

as well as $\langle X_\mu X_\nu - P_\mu P_\nu\rangle$ and eigenvalues $m_{\mu\nu}$ of $X_\mu X_\nu + P_\mu P_\nu$. This leads to the definition of definite attributes $Q_{\mu\nu} := \langle X_\mu X_\nu\rangle$ and $T_{\mu\nu} := \langle P_\mu P_\nu\rangle$ and $R_{\mu\nu}$. It is reasonable to suggest therefore that the characteristics of physically distinct semiclassical limits of a quaplectic system includes the emergence of a quadrupole tensor $Q_{\mu\nu}$ and energy-momentum tensor $T_{\mu\nu}$. The $R_{\mu\nu}$ probably relates to angular momentum.

Chapter 7

Reciprocal Worldline Quantisation

The Hamiltonian quantisation problem is now considered for a reciprocally invariant system consonant with the symmetries of the quaplectic group. The formalism largely follow along the lines of [33] and [34], and the particular arguments closely follow two papers: [35] and [43].

This chapter is organised as follows. In Section 7.1 some notation is defined and the reciprocally invariant action with its associated Lagrangian is introduced. The conjugate momenta operators are defined in Section 7.1.1 and the Noether charges derived in Section 7.1.2. In Section 7.1.3 the notation is redefined in a complex parametrisation with complex variables $z^\mu = \frac{1}{\sqrt{2}}(x^\mu + ip^\mu)$. The Casimir operators are then defined in Section 7.1.4. In Section 7.2, Fock generators are defined in order to diagonalise the Hamiltonian. There are two possible choices for the Fock generators, with the first choice defined in Section 7.2.1 and the second in 7.2.2. The first choice corresponds to the [35] paper and the arguments of that paper are continued in Sections 7.2.1.1 and 7.2.1.2. In the former of these two sections a particular case is studied where the eigenvalue of the conjugate momentum variable Π_θ is taken to be zero. In the latter section the action and Lagrangian are reparametrised in order to perform the worldline quantisation. In Section 7.2.2 the alternative Fock generators are defined and the Hamiltonian is then written in terms of these Fock generators. This section, along with the sections following, closely follows the arguments of the [43] paper. The first class constraint is imposed in Section 7.2.2.1. There are no restrictions placed on the energy-momentum vector p^μ so the mass spectrum is the whole real line with timelike, spacelike, massless and null states. The timelike and massless cases are briefly discussed in Sections 7.2.2.2 and 7.2.2.3 respectively.

7.1 The Action

Recall from Equation 2.46 in Section 2.6 that a general group element of the Heisenberg group $\mathcal{H}(n+1)$ can be realised in terms of abstract translation parameters a^μ, α^μ and c, as the following $(2n+2)\times(2n+2)$ matrices (cf. [57]):

$$g(\alpha^\mu, a^\mu, c) = \begin{pmatrix} I_n & 0 & 0 & \alpha^\mu \\ 0 & I_n & 0 & a^\mu \\ -a_\mu & \alpha_\mu & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that:

$$dg = \begin{pmatrix} 0 & 0 & 0 & d\alpha^\mu \\ 0 & 0 & 0 & da^\mu \\ -da_\mu & d\alpha_\mu & 0 & dc \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The translation parameters can be explicitly stated as space-time and momentum-energy dynamic variables in the usual manner, as follows:

$$\begin{aligned} a^\mu &= (ct, x^i)^T, & \alpha^\mu &= (\frac{e}{c}, p^i)^T \\ a_\mu &= (-ct, x_i), & \alpha_\mu &= (-\frac{e}{c}, p_i) \\ \eta^{\mu\nu} &= \eta_{\mu\nu} = \mathrm{diag}(-++...+) & i &= 1, 2, ..., n. \end{aligned}$$

Contravariant tensors are represented by column vectors and covariant tensors are represented by row vectors. However, as far as the Heisenberg group is concerned, this is an arbitrary choice. An equally valid choice to make would be:

$$\begin{aligned} a^\mu &= (t, \frac{p^i}{b})^T, & \alpha^\mu &= (\frac{e}{b}, -x^i)^T \\ a_\mu &= (-t, \frac{p_i}{b}), & \alpha_\mu &= (\frac{e}{b}, x_i). \end{aligned}$$

There are two other choices that could be made, namely:

$$\begin{aligned} a^\mu &= (\frac{e}{b}, x^i)^T, & \alpha^\mu &= (t, -\frac{p^i}{b})^T \\ a_\mu &= (-\frac{e}{b}, x_i), & \alpha_\mu &= (t, \frac{p_i}{b}), \end{aligned}$$

or finally:

$$\begin{aligned} a^\mu &= (\frac{e}{c}, p^i)^T, & \alpha^\mu &= (ct, -x^i)^T \\ a_\mu &= (-\frac{e}{c}, p_i), & \alpha_\mu &= (ct, x_i). \end{aligned}$$

It will be assumed that the first choice has been made. To reflect this, the abstract parameters α^μ and a^μ will be relabeled as space-time x^μ and energy-momentum

p^μ. However, it should always be kept in mind that the results are equally valid for the other three choices. Furthermore, for ease of notation, the abstract parameter c will be relabeled as 2θ, and the energy-momentum will be written in units of translation: $p^\mu := (\frac{E}{b}, \frac{c}{b}p^i)$.

The line element is given by:

$$\begin{aligned} d\ell^2 &= \tfrac{1}{4}\mathrm{Tr}(g^{-1}dg)^2 \\ &= \tfrac{1}{2}dx^\mu dx_\mu + \tfrac{1}{2}dp^\mu dp_\mu + \tfrac{1}{2}[d\theta - (dx^\mu p_\mu - x^\mu dp_\mu)]^2. \end{aligned}$$

With the introduction of dimensional constants that scale θ with respect to x^μ, and then a further constant $N_0 > 0$ that gives the action the correct units of $\hbar$, the action becomes:

$$S[x^\mu, p^\mu, \theta] = \int d\tau \mathcal{L} \tag{7.1}$$

$$\mathcal{L} = \frac{1}{2N_0}[\dot{x}^2 + \dot{p}^2] + \frac{1}{2N_0}\frac{\alpha}{\lambda_0}[\dot{\theta} - \lambda_0(\dot{x}\cdot p - x\cdot\dot{p})]^2. \tag{7.2}$$

Here, $\alpha\lambda_0 > 0$. The time evolution parameter τ is taken to be dimensionless. This action describes free motion of the variables (x^μ, p^μ) on the Heisenberg group. The variable θ must be there to retain translational invariance.

This action possesses several global symmetries generated by corresponding Noether charges. It has translational invariance in x^μ:

$$x^\mu(\tau) \to x^\mu(\tau) + a^\mu, \qquad p^\mu(\tau) \to p^\mu(\tau), \qquad \theta(\tau) \to \theta(\tau) - \lambda_0 a^\mu p_\mu(\tau). \tag{7.3}$$

The corresponding Noether charges are denoted by $\mathcal{P}^\mu$ and correspond to the system having conservation of total energy-momentum. There is also the dual symmetry: translational invariance in p^μ:

$$x^\mu(\tau) \to x^\mu(\tau), \qquad p^\mu(\tau) \to p^\mu(\tau) + k^\mu, \qquad \theta(\tau) \to \theta(\tau) + \lambda_0 k^\mu x_\mu(\tau). \tag{7.4}$$

The corresponding Noether charges to this symmetry are denoted by $\mathcal{X}^\mu$. There is also translational invariance in θ:

$$x^\mu(\tau) \to x^\mu(\tau), \qquad p^\mu(\tau) \to p^\mu(\tau), \qquad \theta(\tau) \to \theta(\tau) + \theta_0, \tag{7.5}$$

where the corresponding Noether charge is denoted by Q_θ. And so there are three sets of Noether charges associated with translational invariance, $\mathcal{P}^\mu$, $\mathcal{X}^\mu$ and Q_θ.

The system also possesses Lorentz invariance:

$$x^\mu(\tau) \to \Lambda^\mu{}_\nu x^\nu(\tau), \qquad p^\mu(\tau) \to \Lambda^\mu{}_\nu p^\nu(\tau),$$

with

$$\theta(\tau) \to \theta(\tau), \qquad \eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \eta_{\mu\nu},$$

which reduces to the infinitesimal form:

$$\begin{aligned} &x^\mu(\tau) \to x^\mu(\tau) + \omega^\mu{}_\nu x^\nu(\tau), \qquad && p^\mu(\tau) \to p^\mu(\tau) + \omega^\mu{}_\nu p^\nu(\tau), \\ &\theta(\tau) \to \theta(\tau), && \omega_{\mu\nu} = -\omega_{\nu\mu}, \end{aligned} \tag{7.6}$$

where the fact that $\omega_{\mu\nu} = -\omega_{\nu\mu}$ can be determined from considering an infinitesimal Lorentz change in the Lagrangian. The corresponding Noether charges are the total relativistic angular momentum, denoted by $L^{\mu\nu} = -L^{\nu\mu}$.

Finally, the system is invariant under "reciprocal boosts", in infinitesimal form:

$$\begin{aligned} &x^\mu(\tau) \to x^\mu(\tau) + \alpha^\mu{}_\nu p^\nu(\tau), && p^\mu(\tau) \to p^\mu(\tau) - \alpha^\mu{}_\nu x^\nu(\tau), \\ &\theta(\tau) \to \theta(\tau), && \alpha_{\mu\nu} = \alpha_{\nu\mu}, \end{aligned} \tag{7.7}$$

where the fact that $\alpha_{\mu\nu} = \alpha_{\nu\mu}$ can be determined from considering an infinitesimal reciprocal boost change in the Lagrangian. The corresponding Noether charges are denoted by $M^{\mu\nu}$ with $M^{\mu\nu} = M^{\nu\mu}$.

The charges $\mathcal{X}^\mu$, $\mathcal{P}^\mu$ and $\mathcal{Q}_\theta$ are the generators of the algebra of the Heisenberg normal subgroup of the quaplectic group, whereas the charges $L^{\mu\nu}$ and $M^{\mu\nu}$ are the generators of the algebra of the $\mathcal{U}(1,n)$ homogeneous subgroup.

7.1.1 Conjugate Momenta

In the usual Hamiltonian formulation of mechanics, the conjugate momenta can be defined for each velocity as:

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\ .$$

However, the Lagrangian of Equation 7.2 already has momenta coordinates in phase space, which will also have corresponding "conjugate momenta". There are actually nine conjugate "momentum" coordinates, which will be denoted by Π with indices, as such:

$$\begin{aligned} \Pi_\theta &:= \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\alpha}{N_0 \lambda_0}[\dot{\theta} - \lambda_0(\dot{x}\cdot p - x \cdot \dot{p})] \\ \Pi_x{}^\mu &:= \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} = \frac{1}{N_0}\dot{x}^\mu - \lambda_0 \Pi_\theta p^\mu \\ \Pi_p{}^\mu &:= \frac{\partial \mathcal{L}}{\partial \dot{p}_\mu} = \frac{1}{N_0}\dot{p}^\mu + \lambda_0 \Pi_\theta x^\mu \end{aligned}$$

so that there are four Π_x conjugate momenta and another four Π_p.

In the canonically quantized system, these conjugate momentum variables are operators with commutation relations given by the corresponding brackets multiplied by i:

$$[x^\mu, \Pi_x{}^\nu] = i\hbar\eta^{\mu\nu}, \qquad [p^\mu, \Pi_p{}^\nu] = i\hbar\eta^{\mu\nu}, \qquad [\theta, \Pi_\theta] = i\hbar.$$

Thus the quantised dynamics is realised by a tensor product of Heisenberg algebras [35].

7.1.2 Global Symmetries and Noether Charges

From the chain rule,

$$\begin{aligned}\delta\mathcal{L} =&\frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta\dot{x}^\mu + \frac{\partial\mathcal{L}}{\partial p^\mu}\delta p^\mu + \frac{\partial\mathcal{L}}{\partial\dot{p}^\mu}\delta\dot{p}^\mu + \frac{\partial\mathcal{L}}{\partial\theta}\delta\theta + \frac{\partial\mathcal{L}}{\partial\dot{\theta}}\delta\dot{\theta}\\ =&\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\right)\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta\dot{x}^\mu + \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{p}^\mu}\right)\delta p^\mu + \frac{\partial\mathcal{L}}{\partial\dot{p}^\mu}\delta\dot{p}^\mu +\\ &+\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right)\delta\theta + \frac{\partial\mathcal{L}}{\partial\dot{\theta}}\delta\dot{\theta},\end{aligned} \tag{7.8}$$

where the Euler-Lagrange equation has been used to obtain the second line.

The symmetry described by Equation 7.5 clearly has a corresponding Noether charge that is proportional to Π_θ itself. Therefore it will be chosen to be:

$$\mathcal{Q}_\theta := \Pi_\theta.$$

The symmetry described by Equation 7.3 corresponds to:

$$\delta\mathcal{L} = a^\mu\frac{d}{d\tau}\left(\Pi_{x\mu} - \lambda_0 p_\mu\Pi_\theta\right),$$

since $\frac{d}{d\tau}a^\mu = 0$. The corresponding Noether charge for this symmetry is proportional to the quantity in parentheses. It will thus be chosen to be:

$$\mathcal{P}_\mu := \frac{1}{\sqrt{2}}\left(\Pi_{x\mu} - \lambda_0\Pi_\theta p_\mu\right). \tag{7.9}$$

A similar calculation shows that the Noether charge corresponding to invariance under the symmetry described by Equation 7.4 is given by:

$$\mathcal{X}_\mu := \frac{1}{\sqrt{2}}\left(\Pi_{p_\mu} + \lambda_0\Pi_\theta x_\mu\right). \tag{7.10}$$

These three Noether charges together obey the Heisenberg commutation relations:

$$[\mathcal{X}^\mu, \mathcal{P}^\nu] = i\hbar\lambda_0\eta^{\mu\nu}\mathcal{Q}_\theta.$$

A Lorentz transformation described by Equation 7.6 corresponds to an infinitesimal change in the Lagrangian of:

$$\begin{aligned}\delta\mathcal{L} &= \omega_{\mu\nu}\left(\frac{d}{d\tau}\left(\Pi_x{}^\mu\right)x^\nu + \Pi_x{}^\mu\frac{d}{d\tau}\left(x^\nu\right) + \frac{d}{d\tau}\left(\Pi_p{}^\mu\right)p^\nu + \Pi_p{}^\mu\frac{d}{d\tau}\left(p^\nu\right)\right)\\ &= \omega_{\mu\nu}\frac{d}{d\tau}\left(\Pi_x{}^\mu x^\nu + \Pi_p{}^\mu p^\nu\right),\end{aligned}$$

and so the Noether charge associated with Lorentz invariance is given by the quantity in parentheses, or equivalently by:

$$L^{\mu\nu} := x^\mu\Pi_x{}^\nu - x^\nu\Pi_x{}^\mu + p^\mu\Pi_p{}^\nu - p^\nu\Pi_p{}^\mu. \tag{7.11}$$

A reciprocal boost transformation described by Equation 7.7 corresponds to an infinitesimal change in the Lagrangian of:

$$\begin{aligned}\delta\mathcal{L} &= \alpha_{\mu\nu}\left(\left[\frac{d}{d\tau}\left(\Pi_x{}^{\mu}\right)p^{\nu} + \Pi_x{}^{\mu}\frac{d}{d\tau}\left(p^{\nu}\right)\right] - \left[\frac{d}{d\tau}\left(\Pi_p{}^{\mu}\right)x^{\nu} + \Pi_p{}^{\mu}\frac{d}{d\tau}\left(x^{\nu}\right)\right]\right) \\ &= \alpha_{\mu\nu}\frac{d}{d\tau}\left(\Pi_x{}^{\mu}p^{\nu} - \Pi_p{}^{\mu}x^{\nu}\right)\end{aligned}$$

and so the Noether charge associated with reciprocal boost invariance is given by the quantity in parentheses, or equivalently by:

$$M^{\mu\nu} := \left(p^{\mu}\Pi_x{}^{\nu} + p^{\nu}\Pi_x{}^{\mu}\right) - \left(x^{\mu}\Pi_p{}^{\nu} + x^{\nu}\Pi_p{}^{\mu}\right) \tag{7.12}$$

By introducing auxiliary variables:

$$\mathbb{P}^{\mu} := \frac{1}{\sqrt{2}}\left(\Pi_x{}^{\mu} + \lambda_0\Pi_\theta p^{\mu}\right) \qquad \mathbb{X}^{\mu} := \frac{1}{\sqrt{2}}\left(\Pi_p{}^{\mu} - \lambda_0\Pi_\theta x^{\mu}\right), \tag{7.13}$$

that obey the commutation relation:

$$\left[\mathbb{X}^{\mu}, \mathbb{P}^{\nu}\right] = -i\hbar\lambda_0\eta^{\mu\nu}$$

the Noether charges can otherwise be written as:

$$\begin{aligned}L^{\mu\nu} &= x^{\mu}\mathcal{P}^{\nu} - x^{\nu}\mathcal{P}^{\mu} + p^{\mu}\mathbb{X}^{\nu} - p^{\nu}\mathbb{X}^{\mu} \\ M^{\mu\nu} &= \frac{1}{2}\left(p^{\mu}\mathcal{P}^{\nu} + p^{\mu}\mathbb{P}^{\nu}\right) - \frac{1}{2}\left(x^{\mu}\mathcal{X}^{\nu} + x^{\mu}\mathbb{X}^{\nu}\right) + (\mu \leftrightarrow \nu).\end{aligned}$$

The total angular-momentum $L^{\mu\nu}$ can be broken up into two parts:

$$L^{\mu\nu} = L^{\mu\nu}_{\text{orbital}} + L^{\mu\nu}_{\text{internal}},$$

where $L^{\mu\nu}_{\text{orbital}} = x^{\mu}\mathcal{P}^{\nu} - x^{\nu}\mathcal{P}^{\mu}$ is the orbital angular momentum. The separation of the space-time degrees of freedom x^{μ} from the dual variables p^{μ} shows that the space-time degrees of freedom describe the position of the reciprocal particle in Minkowski space, whereas the dual variables play the role of internal degrees of freedom of the particle.

A third way exists of expressing the $L^{\mu\nu}$ and $M^{\mu\nu}$ Noether charges. This notation explicitly separates the Noether charges associated with translational invariance from the auxiliary variables defined in Equation 7.13. Provided $\lambda_0\Pi_\theta \neq 0$,

$$L^{\mu\nu} = \frac{1}{\lambda_0\Pi_\theta}\left[\left(\mathcal{X}^{\mu}\mathcal{P}^{\nu} - \mathcal{X}^{\nu}\mathcal{P}^{\mu}\right) - \left(\mathbb{X}^{\mu}\mathbb{P}^{\nu} - \mathbb{X}^{\nu}\mathbb{P}^{\mu}\right)\right] \tag{7.14}$$

$$M^{\mu\nu} = \frac{1}{\lambda_0\Pi_\theta}\left[\left(\mathcal{X}^{\mu}\mathcal{X}^{\nu} + \mathcal{P}^{\nu}\mathcal{P}^{\mu}\right) - \left(\mathbb{X}^{\mu}\mathbb{X}^{\nu} + \mathbb{P}^{\nu}\mathbb{P}^{\mu}\right)\right]. \tag{7.15}$$

The evaluation of the commutation relations of the Noether charges is straightforward. The nonvanishing brackets are:

$$\begin{aligned}
[L^{\mu\nu}, \mathcal{X}^\rho] &= i\hbar \left(\eta^{\mu\rho}\mathcal{X}^\nu - \eta^{\nu\rho}\mathcal{X}^\mu\right), \\
[L^{\mu\nu}, \mathcal{P}^\rho] &= i\hbar \left(\eta^{\mu\rho}\mathcal{P}^\nu - \eta^{\nu\rho}\mathcal{P}^\mu\right) \\
[M^{\mu\nu}, \mathcal{X}^\rho] &= i\hbar \left(\eta^{\mu\rho}\mathcal{P}^\nu + \eta^{\nu\rho}\mathcal{P}^\mu\right) \\
[M^{\mu\nu}, \mathcal{P}^\rho] &= -i\hbar \left(\eta^{\mu\rho}\mathcal{X}^\nu + \eta^{\nu\rho}\mathcal{X}^\mu\right) \\
[L^{\mu\nu}, L^{\rho\sigma}] &= i\hbar \left(\eta^{\mu\rho}L^{\nu\sigma} - \eta^{\mu\sigma}L^{\nu\rho} - \eta^{\nu\rho}L^{\mu\sigma} + \eta^{\nu\sigma}L^{\mu\rho}\right) \\
[L^{\mu\nu}, M^{\rho\sigma}] &= i\hbar \left(\eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}\right) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= i\hbar \left(\eta^{\mu\rho}L^{\nu\sigma} + \eta^{\mu\sigma}L^{\nu\rho} + \eta^{\nu\rho}L^{\mu\sigma} + \eta^{\nu\sigma}L^{\mu\rho}\right).
\end{aligned}$$

This is the Lie algebra of the quaplectic group (cf. Equations 3.29 and [55]).

The final global symmetry of the system is the invariance under evolution of the time parameter:

$$\tau \to \tau + \tau_0. \tag{7.16}$$

The conserved Noether charge associated with this symmetry is the canonical Hamiltonian H_0. This is given by:

$$H_0 = \frac{N_0}{2}\left(\mathbb{P}^2 + \mathbb{X}^2 + \frac{\lambda_0}{\alpha}{\Pi_\theta}^2\right). \tag{7.17}$$

7.1.3 Complex Parametrisation

Introducing a complex parametrisation simplifies the notation. The configuration space variables become:

$$z^\mu := \frac{1}{\sqrt{2}}[x^\mu + ip^\mu].$$

The action is now a function of z^μ, $\overline{z}^\mu$ and θ, where a bar above a quantity denotes the complex conjugate of that quantity. The Lagrangian (Equation 7.2) becomes:

$$\mathcal{L} = \frac{1}{N_0}\dot{\overline{z}}\cdot\dot{z} + \frac{1}{2N_0}\frac{\alpha}{\lambda_0}\left[\dot{\theta} + i\lambda_0\left(\dot{\overline{z}}\cdot z - \overline{z}\cdot\dot{z}\right)\right]^2.$$

The global symmetries of Equations 7.3, 7.4, 7.5, 7.6 and 7.7 can be expressed in infinitesimal form as:

$$z^\mu(\tau) \to z^\mu(\tau) + \Omega^\mu{}_\nu z^\nu(\tau) + A^\mu, \tag{7.18}$$

$$\theta(\tau) \to \theta(\tau) + \theta_0 - i\lambda_0\left(\overline{z}(\tau)\cdot A - z(\tau)\cdot\overline{A}\right), \tag{7.19}$$

and:

$$\Omega_{\mu\nu} := \omega_{\mu\nu} - i\alpha_{\mu\nu}, \qquad \overline{\Omega}_{\mu\nu} = -\Omega_{\nu\mu}, \qquad A^\mu := \frac{1}{\sqrt{2}}\left(a^\mu + ik^\mu\right).$$

The conjugate momenta can be defined as before:

$$\begin{aligned}\Pi^{\mu} :&= \frac{\partial \mathcal{L}}{\partial \dot{z}_{\mu}} \\ &= \frac{1}{N_0}\dot{\overline{z}}^{\mu} - \lambda_0 \Pi_{\theta} i \overline{z} \\ &= \frac{1}{\sqrt{2}}\left[\left(\frac{1}{N_0}\dot{x}^{\mu} - \lambda_0 \Pi_{\theta} p^{\mu}\right) - i\left(\frac{1}{N_0}\dot{p}^{\mu} + \lambda_0 \Pi_{\theta} x^{\mu}\right)\right] \\ &= \frac{1}{\sqrt{2}}\left(\Pi_x{}^{\mu} - i\Pi_p{}^{\mu}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{\Pi}^{\mu} :&= \frac{\partial \mathcal{L}}{\partial \dot{\overline{z}}_{\mu}} \\ &= \frac{1}{\sqrt{2}}\left(\Pi_x{}^{\mu} + i\Pi_p{}^{\mu}\right).\end{aligned}$$

These conjugate momenta obey the following Heisenberg commutation relations:

$$\left[z^{\mu}, \Pi^{\nu}\right] = i\hbar\eta^{\mu\nu} \qquad \left[\overline{z}^{\mu}, \overline{\Pi}^{\nu}\right] = i\hbar\eta^{\mu\nu}.$$

The Noether charges can be evaluated in complex form as before, from the complex variation of Equation 7.8:

$$\begin{aligned}\delta\mathcal{L} =& \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}_{\mu}}\right)\delta z^{\mu} + \frac{\partial \mathcal{L}}{\partial \dot{z}_{\mu}}\delta\dot{z}^{\mu} + \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\overline{z}}_{\mu}}\right)\delta\overline{z}^{\mu} + \frac{\partial \mathcal{L}}{\partial \dot{\overline{z}}_{\mu}}\delta\dot{\overline{z}}^{\mu} + \\ &+ \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right)\delta\theta + \frac{\partial \mathcal{L}}{\partial \dot{\theta}}\delta\dot{\theta}.\end{aligned}$$

The first-order Noether charges can be found by a simple calculation. They are:

$$\mathcal{Z}^{\mu} := \frac{1}{\sqrt{2}}\left(\mathcal{X}^{\mu} + i\mathcal{P}^{\mu}\right) = \frac{1}{\sqrt{2}}\left(i\Pi^{\mu} + \lambda_0 \Pi_{\theta}\overline{z}^{\mu}\right) \tag{7.20}$$

$$\overline{\mathcal{Z}}^{\mu} := \frac{1}{\sqrt{2}}\left(\mathcal{X}^{\mu} - i\mathcal{P}^{\mu}\right) = \frac{1}{\sqrt{2}}\left(-i\overline{\Pi}^{\mu} + \lambda_0 \Pi_{\theta} z^{\mu}\right) \tag{7.21}$$

For the symmetry described in Equations 7.18 and 7.19, the infinitesimal Lagrangian becomes:

$$\begin{aligned}\delta\mathcal{L} =& \dot{\Pi}_{\mu}\left(\Omega^{\mu}{}_{\nu} z^{\nu} + A^{\mu}\right) + \Pi_{\mu}\Omega^{\mu}{}_{\nu}\dot{z}^{\nu} + \dot{\overline{\Pi}}_{\mu}\left(-\Omega_{\nu}{}^{\mu}\overline{z}^{\nu} + \overline{A}^{\mu}\right) - \overline{\Pi}_{\mu}\Omega_{\nu}{}^{\mu}\dot{\overline{z}}^{\nu} - \\ &- \Pi_{\theta} i\lambda_0\left(\dot{\overline{z}}\cdot A - \dot{z}\cdot\overline{A}\right) \\ =& \Omega_{\mu\nu}\left(z^{\mu}\dot{\Pi}^{\nu} + \dot{z}^{\mu}\Pi^{\nu} - \overline{z}^{\nu}\dot{\overline{\Pi}}^{\mu} - \dot{\overline{z}}^{\nu}\overline{\Pi}^{\mu}\right) + \dot{\Pi}\cdot A + \dot{\overline{\Pi}}\cdot\overline{A} - i\Pi_{\theta}\lambda_0\left(\dot{\overline{z}}\cdot A - \dot{z}\cdot\overline{A}\right) \\ =& \Omega_{\mu\nu}\frac{d}{d\tau}\left(z^{\mu}\Pi^{\nu} - \overline{z}^{\nu}\overline{\Pi}^{\mu}\right) + \left(\dot{\Pi} - i\lambda_0\Pi_{\theta}\dot{\overline{z}}\right)\cdot A + \left(\dot{\overline{\Pi}} + i\lambda_0\Pi_{\theta}\dot{z}\right)\cdot\overline{A}.\end{aligned}$$

The second and third terms of the final line both vanish, since the quantities in parentheses are the time derivatives of the first-order Noether charges, which

of course are constant in time. Therefore, the second-order Noether charge is proportional to the quantity in parentheses in the first term:

$$E^{\mu\nu} = -i\left(z^\mu \Pi^\nu - \overline{z}^\nu \overline{\Pi}^\mu\right) \tag{7.22}$$

$$= \frac{1}{2}\left(M^{\mu\nu} - iL^{\mu\nu}\right). \tag{7.23}$$

The final Noether charge remains unchanged:

$$\mathcal{Q}_\theta := \Pi_\theta. \tag{7.24}$$

Introducing again the auxiliary, complex, variables:

$$\mathbb{Z}^\mu := \frac{1}{\sqrt{2}}\left(\mathbb{X}^\mu + i\mathbb{P}^\mu\right) = \frac{1}{\sqrt{2}}\left(i\Pi^\mu - \lambda_0 \Pi_\theta \overline{z}^\mu\right) \tag{7.25}$$

$$\overline{\mathbb{Z}}^\mu := \frac{1}{\sqrt{2}}\left(\mathbb{X}^\mu - i\mathbb{P}^\mu\right) = \frac{1}{\sqrt{2}}\left(-i\overline{\Pi}^\mu - \lambda_0 \Pi_\theta z^\mu\right), \tag{7.26}$$

allows the second-order Noether charges to be written as:

$$\begin{aligned} E^{\mu\nu} &= \frac{1}{2}\left(z^\mu \mathcal{Z}^\nu - \overline{z}^\nu \overline{\mathcal{Z}}^\mu\right) + \frac{1}{2}\left(z^\mu \mathbb{Z}^\nu - \overline{z}^\nu \overline{\mathbb{Z}}^\mu\right) \\ &= -i\frac{1}{2\lambda_0 \Pi_\theta}\left(\overline{\mathcal{Z}}^\mu \mathcal{Z}^\nu - \overline{\mathbb{Z}}^\mu \mathbb{Z}^\nu\right), \end{aligned}$$

where the second line is only true if $\lambda_0 \Pi_\theta \neq 0$.

The non-vanishing commutation relations of the complex Noether charges are:

$$\begin{aligned} \left[\mathcal{Z}^\mu, \overline{\mathcal{Z}}^\nu\right] &= \hbar\lambda_0 \eta^{\mu\nu} \mathcal{Q}_\theta \\ \left[E^{\mu\nu}, \mathcal{Z}^\rho\right] &= \hbar\eta^{\mu\rho} \mathcal{Z}^\nu \\ \left[E^{\mu\nu}, \overline{\mathcal{Z}}^\rho\right] &= -\hbar\eta^{\nu\rho} \overline{\mathcal{Z}}^\mu \\ \left[E^{\mu\nu}, E^{\rho\sigma}\right] &= \hbar\left(\eta^{\mu\sigma} E^{\rho\nu} - \eta^{\nu\rho} E^{\mu\sigma}\right). \end{aligned}$$

Additionally, the auxiliary variables satisfy the Heisenberg commutation relations:

$$\left[\mathbb{Z}^\mu, \overline{\mathbb{Z}}^\nu\right] = -\hbar\lambda_0 \eta^{\mu\nu} \mathcal{Q}_\theta \tag{7.27}$$

while all combinations of these variables with the complex Noether charges $\mathcal{Z}^\mu$ and $\overline{\mathcal{Z}}^\mu$ commute.

The first-order Noether charges $\mathcal{Z}^\mu$, together with $\mathcal{Q}_\theta$, generate the algebra of the Heisenberg group $\mathcal{H}(n+1)$. The second-order Noether charges $E^{\mu\nu}$ generate the algebra of the pseudo-unitary group $\mathcal{U}(n,1)$. Together, these Noether charges generate the algebra of the quaplectic group $\mathcal{Q}(1,n) := \mathcal{U}(1,n) \ltimes \mathcal{H}(n+1)$.

The canonical Hamiltonian of Equation 7.17 can be expressed succinctly in terms of the auxiliary variables of Equation 7.25. Since $\mathbb{P}^2 + \mathbb{X}^2 = \overline{\mathbb{Z}} \cdot \mathbb{Z} + \mathbb{Z} \cdot \overline{\mathbb{Z}}$:

$$H_0 = \frac{N_0}{2}\left(\overline{\mathbb{Z}} \cdot \mathbb{Z} + \mathbb{Z} \cdot \overline{\mathbb{Z}} + \frac{\lambda_0}{\alpha}{\Pi_\theta}^2\right). \tag{7.28}$$

Since the first-order complex Noether charges $\mathcal{Z}^\mu$ commute with $\mathbb{Z}^\mu$ and $\overline{\mathbb{Z}}^\mu$, this form of the Hamiltonian shows explicitly that the first-order Noether charges are conserved. The verification of the fact that the second-order Noether charges $E^{\mu\nu}$ are conserved follows from a simple calculation.

7.1.4 Casimir Operators

The $\mathcal{U}(1,n)$ algebra possesses the Casimir operator:

$$\begin{aligned} C :&= E^{\mu}{}_{\mu} = \frac{1}{2} M^{\mu}{}_{\mu} = p \cdot \Pi_x - x \cdot \Pi_p \\ &= \overline{C}. \end{aligned} \tag{7.29}$$

While this Casimir operator commutes with the second-order Noether charges, it does not commute with the first-order Noether charges associated with the Heisenberg algebra:

$$[C, \mathcal{Z}^{\mu}] = \hbar \mathcal{Z}^{\mu}, \qquad \left[C, \overline{\mathcal{Z}}^{\mu}\right] = -\hbar \overline{\mathcal{Z}}^{\mu},$$

or with the real form of the Noether charges,

$$[C, \mathcal{X}^{\mu}] = i\hbar \mathcal{P}^{\mu}, \qquad [C, \mathcal{P}^{\mu}] = -i\hbar \mathcal{X}^{\mu}.$$

At the end of Section 7.1.3, it was stated that Equation 7.28 explicitly shows that the first-order Noether charges are conserved. In fact, the canonical Hamiltonian of Equation 7.28 can be expressed in terms of these first-order complex Noether charges as follows:

$$H_0 = \frac{1}{2} \left(\overline{\mathcal{Z}} \cdot \mathcal{Z} + \mathcal{Z} \cdot \overline{\mathcal{Z}} + 4\lambda_0 \Pi_\theta C + \frac{\lambda_0}{\alpha} {\Pi_\theta}^2 \right), \tag{7.30}$$

where clearly, $\overline{\mathbb{Z}} \cdot \mathbb{Z} + \mathbb{Z} \cdot \overline{\mathbb{Z}} = \overline{\mathcal{Z}} \cdot \mathcal{Z} + \mathcal{Z} \cdot \overline{\mathcal{Z}} + 4\lambda_0 \Pi_\theta C$. This implies that another Casimir operator, which is actually the linear Casimir operator and will be denoted by C_1, is given by:

$$C_1 = \overline{\mathbb{Z}} \cdot \mathbb{Z} + \mathbb{Z} \cdot \overline{\mathbb{Z}} = \mathbb{P}^2 + \mathbb{X}^2. \tag{7.31}$$

The linear Casimir operator is thus defined purely in terms of the auxiliary generators. In terms of the first-order Noether charges and the first Casimir operator - which will be relabeled as C_2 - of Equation 7.29, the linear Casimir operator is given by:

$$\begin{aligned} C_1 &= \overline{\mathcal{Z}} \cdot \mathcal{Z} + \mathcal{Z} \cdot \overline{\mathcal{Z}} + 4\lambda_0 \Pi_\theta C_2 \\ &= 2\left(\mathcal{P}^2 + \mathcal{X}^2\right) + 4\lambda_0 \Pi_\theta C_2. \end{aligned}$$

Conversely,

$$C_2 = \frac{1}{4\lambda_0 \Pi_\theta} \left(C_1 - 2\mathcal{P}^2 - 2\mathcal{X}^2 \right).$$

7.2 Quantum Spectrum

In this section, Fock generators will be defined in order to diagonalise the Hamiltonian and thus determine the quantum spectrum. There are two choices available for the sets of Fock generators. In the first choice the Fock vacuum is invariant under the action of $L^{\mu\nu}$ (Lorentz invariance) and so Lorentz covariance of all quantum states is ensured throughout, however any state involving an odd power of the creation operator $a_s^{0\dagger}$ $s = +, -$ (see Section 7.2.1 for the definition of these Fock generators) is of negative norm due to the negative signature of the time component of the Minkowski metric. This is the case studied in [35] and in 7.2.1. The second choice removes the problem of negative norm states but sacrifices Lorentz invariance of the Fock vacuum under the action of $L_{\mu\nu}$. This is the case studied in [43] and in 7.2.2.

7.2.1 Lorentz Covariant Fock Generators

Consider eigenstates $|\ell\rangle$ of the operator Π_θ:

$$\Pi_\theta |\ell\rangle = \pi_\theta |\ell\rangle = \hbar(\ell + \lambda) |\ell\rangle ,$$

where π_θ is the abstract eigenvalue of Π_θ and this eigenvalue is written explicitly as $\hbar(\ell + \lambda)$ in the second equation.

In the first choice of Fock algebra generators, all generators with daggers are creation operators whereas those without daggers are annihilation operators. In this framework there are two possible sets of Fock algebra generators that can be defined, depending on the sign of the product: $\lambda_0\pi_\theta$. These operators are proportional to $\mathbb{Q}^\mu$ and $\overline{\mathbb{Q}}^\mu$ or to $\mathcal{Q}^\mu$ and $\overline{\mathcal{Q}}^\mu$. They are given by:

$$a_+^\mu := \sqrt{\frac{|\lambda_0\pi_\theta|}{2\hbar}} \left(\overline{z}^\mu + i \frac{1}{|\lambda_0\pi_\theta|} \pi^\mu \right), \quad a_+^{\mu\,\dagger} := \sqrt{\frac{|\lambda_0\pi_\theta|}{2\hbar}} \left(z^\mu - i \frac{1}{|\lambda_0\pi_\theta|} \overline{\Pi}^\mu \right) \tag{7.32}$$

$$a_-^\mu := \sqrt{\frac{|\lambda_0\pi_\theta|}{2\hbar}} \left(z^\mu + i \frac{1}{|\lambda_0\pi_\theta|} \overline{\Pi}^\mu \right), \quad a_-^{\mu\,\dagger} := \sqrt{\frac{|\lambda_0\pi_\theta|}{2\hbar}} \left(\overline{z}^\mu - i \frac{1}{|\lambda_0\pi_\theta|} \Pi^\mu \right). \tag{7.33}$$

This defines two mutually commuting sets of Fock algebras for each spacetime component $\mu = 0, 1, ..., n$. The constant of proportionality has been chosen such that, for $s = +, -$, the following holds:

$$\left[a_s^\mu, a_{s'}^{\nu\,\dagger} \right] = \delta_{s,s'} \eta^{\mu\nu}.$$

The Noether charges of Equation 7.20 and the complex auxiliary variables of Equation 7.25 can now be written in terms of these Fock operators. However, the form of these variables depends on the sign of the product $\lambda_0\pi_\theta$. Therefore, with $\eta = +, -$ defined to be the sign of this product, the variables are given by:

$$\begin{aligned} &\text{if } \eta = + : \qquad \mathcal{Z}^\mu = \sqrt{2\hbar\,|\lambda_0\pi_\theta|}(-ia_+^\mu) \qquad \mathbb{Z}^\mu = \sqrt{2\hbar\,|\lambda_0\pi_\theta|}(ia_-^{\mu\,\dagger}) \\ &\text{if } \eta = - : \qquad \mathcal{Z}^\mu = \sqrt{2\hbar\,|\lambda_0\pi_\theta|}(ia_-^{\mu\,\dagger}) \qquad \mathbb{Z}^\mu = \sqrt{2\hbar\,|\lambda_0\pi_\theta|}(-ia_+^\mu). \end{aligned}$$

Since $\Pi^\mu = \frac{1}{2}\left(\mathcal{Z}^\mu + \mathbb{Z}^\mu\right)$ from Equations 7.20 and 7.25, this means that Π^μ can also be written in terms of these Fock operators, as:

$$\Pi^\mu = i\sqrt{\frac{\hbar\left|\lambda_0\pi_\theta\right|}{2}}\left(a_-^{\mu\,\dagger} - a_+^\mu\right).$$

The Noether charges can also be written in terms of Fock operators, as:

$$\mathcal{P}^\mu = i\sqrt{\frac{\hbar\left|\lambda_0\pi_\theta\right|}{2}}\left(a_\eta^{\mu\dagger} - a_\eta^\mu\right) \qquad \mathcal{X}^\mu = \eta\sqrt{\frac{\hbar\left|\lambda_0\pi_\theta\right|}{2}}\left(a_\eta^{\mu\dagger} + a_\eta^\mu\right).$$

The total angular momentum, of the form of Equation 7.14, can be rewritten as:

$$L^{\mu\nu} = \frac{1}{\lambda_0\pi_\theta}\left[\left(\mathcal{X}^\mu\mathcal{P}^\nu - \mathcal{X}^\nu\mathcal{P}^\mu\right) - i\frac{\hbar}{2}\left({a_{-\eta}^\mu}^\dagger a_{-\eta}^\nu - {a_{-\eta}^\nu}^\dagger a_{-\eta}^\mu\right)\right]. \tag{7.34}$$

The Fock operators allow the Hamiltonian to be diagonalised. In terms of Fock operators, the Hamiltonian is given by:

$$\begin{aligned} H_0 &= 2\hbar N_0\left|\lambda_0\pi_\theta\right| a_{-\eta}^\dagger\cdot a_{-\eta} + \frac{1}{2}N_0\frac{\lambda_0}{\alpha}{\pi_\theta}^2 + \hbar N_0\left|\lambda_0\pi_\theta\right|(n+1) \\ &= \hbar N_0\left|\lambda_0\pi_\theta\right|\left(2a_{-\eta}^\dagger\cdot a_{-\eta} + n + 1\right) + \frac{1}{2}N_0\frac{\lambda_0}{\alpha}{\pi_\theta}^2. \end{aligned}$$

In the expression for the total angular momentum of Equation 7.34, it can be seen that only the operators $a_{-\eta}^\mu$, ${a_{-\eta}^\mu}^\dagger$ make contributions to the Pauli-Lubanksi operator and thus to the internal spin representation of quantum states. Therefore these $a_{-\eta}^\mu$, ${a_{-\eta}^\mu}^\dagger$ operators are to be interpreted as defining the internal degrees of freedom whereas the $\mathcal{X}^\mu$, $\mathcal{P}^\mu$ terms provide the orbital angular momentum degrees of freedom.

7.2.1.1 The Particular Case $\Pi_\theta = 0$

In the particular case of vanishing eigenvalue π_θ of the operator Π_θ, most quantities are greatly simplified. In particular:

$$\mathcal{Z}^\mu = \mathbb{Z}^\mu = \Pi^\mu, \quad \mathcal{P}^\mu = \mathbb{P}^\mu = \Pi_x^\mu, \quad \mathcal{X}^\mu = \mathbb{X}^\mu = \Pi_p^\mu.$$

An immediate consequence is that:

$$\left[\mathcal{X}^\mu, \mathcal{P}^\nu\right] = 0.$$

The Hamiltonian becomes:

$$H = \tfrac{1}{2}N_0\left[\mathcal{P}^2 + \mathcal{X}^2\right],$$

and the angular momentum operator $L^{\mu\nu}$ becomes:

$$\begin{aligned} L^{\mu\nu} &= x^\mu\Pi_x^\nu - x^\nu\Pi_x^\mu + p^\mu\Pi_p^\nu - x^\nu\Pi_p^\mu \\ &= x^\mu\mathcal{P}^\nu - x^\nu\mathcal{P}^\mu + p^\mu\mathcal{X}^\nu - p^\nu\mathcal{X}^\mu \\ &= L_{\text{orbital}}^{\mu\nu} + p^\mu\mathcal{X}^\nu - p^\nu\mathcal{X}^\mu, \end{aligned}$$

so that the sector (x^μ, Π_x^μ) takes the role of the external or orbital degrees of freedom whereas the sector (p^μ, Π_p^μ) takes the role of the internal degrees of freedom. Diagonalisation can be achieved in the conjugate momentum basis which is equivalent to the basis of $(\mathcal{P}^\mu, \mathcal{X}^\mu)$ in this case. There are no negative norm states, but $\mathcal{P}^2$ and $\mathcal{X}^2$ can take arbitrary negative values and therefore the Hamiltonian can as well.

7.2.1.2 Worldline Quantisation

The action and Lagrangian can be reparametrised in a diffeomorphic-invariant manner. See [33], [34] and [44]. The action becomes:

$$S[x^\mu, p^\mu, \theta; e] = \int L[x^\mu, p^\mu, \theta; e] d\tau,$$

where L is the reparametrised Lagrangian. To reparametrise the Lagrangian, the einbein is introduced into the action. The einbein, $e(\tau)$ is the square-root of the metric on the world-line, and is dimensionless. Mathematically, it is given by:

$$e(\tau) = \frac{dt(\tau)}{d\tau}.$$

Note that e^2 is indeed the world-line metric.

The reparametrised Lagrangian is:

$$L = \frac{1}{e}\mathcal{L} + e\Lambda. \tag{7.35}$$

where $\mathcal{L}$ is the Lagrangian of Equation 7.2 and Λ is a real constant with the same dimensions as the original Lagrangian, taking the role of a cosmological constant on the worldline. The action is therefore:

$$S[x^\mu, p^\mu, \theta; e] = \int d\tau \left[\frac{1}{e}\mathcal{L} + e\Lambda\right]. \tag{7.36}$$

This action can be seen to be the same as the original action, by looking at the Euler Lagrange equation for $\mu := \frac{1}{e}$:

$$\frac{\partial}{\partial\tau}\frac{\partial L}{\partial\dot{\mu}} - \frac{\partial L}{\partial\mu} = 0.$$

This yields $\mu^2 = \Lambda/\mathcal{L}$ and so therefore $e^2 = \mathcal{L}/\Lambda$. Substituting this into Equation 7.35 yields the original Lagrangian of Equation 7.2 up to a power and multiplied by a constant. Specifically, it is equal to

$$L = 2\sqrt{\mathcal{L}\Lambda}.$$

In the Hamiltonian formulation of the system, the einbein is a Lagrange multiplier for a constraint Φ on phase-space; namely the constraint:

$$\begin{aligned} \Phi &= H - \Lambda = 0 \\ \Rightarrow H &= \Lambda, \end{aligned}$$

and the action is given by:

$$S = \int d\tau \left[\dot{x} \cdot \Pi_x + \dot{p} \cdot \Pi_p + \dot{\theta}\Pi_\theta - e(H - \Lambda) \right].$$

All previously defined Noether charges commute with the Hamiltonian, and so the representations of the quaplectic group correspond to quantum physical states. An appropriate choice for the cosmological constant Λ would project out physical states of negative norm.

In any Π_θ superselection sector, where the eigenvalue π_θ of Π_θ is not equal to 0, the eigenspectrum of the Hamiltonian is given by:

$$H = N_0 \frac{\lambda_0}{2\alpha} \hbar^2 (\ell + \lambda)^2 + \hbar N_0 \left| \lambda_0 \hbar (\ell + \lambda) \right| (2N + n + 1).$$

There are no negative norm states present only in the Fock vacuum state $N = 0$, however, given a choice of values for the parameters λ_0 and α there are always a range of possible values for Λ such that N can be equal to 0 for some unique choice of λ and ℓ, with the gauge invariance constraint projecting out negative norm states [35]. The spectrum of $\mathcal{P}^\mu$ can take arbitrary values which are representations of the Poincaré group with vanishing spin. However, the mass spectrum is then the real line, with $(Mc)^2 = -\mathcal{P}^2$. There will therefore be timelike states, massless states and spacelike states.

In the particular superselection sector where the eigenvalue π_θ of Π_θ is 0, there are no negative norm states and the gauge invariance constraint reduces to:

$$\begin{aligned} -(Mc)^2 &= \mathcal{P}^2 \\ &= \frac{2}{N_0}\Lambda - \mathcal{X}^2. \end{aligned}$$

But again, there are no restrictions on the spectra of $\mathcal{X}^\mu$ and $\mathcal{X}^2$, so that the mass spectrum is the whole real line including timelike, massless and spacelike branches.

7.2.2 Alternative Choice of Fock Generators

An alternative choice of Fock algebra generators exists which eliminates the existence of negative norm states but also sacrifices Lorentz invariance of the Fock vacuum state. Instead of the choices given by Equation 7.32, consider instead the Fock algebra generators:

$$\begin{pmatrix} a \\ b_1^\dagger \\ b_2^\dagger \\ b_3^\dagger \end{pmatrix} \propto \begin{pmatrix} \mathbb{Z}_0 \\ \mathbb{Z}_1 \\ \mathbb{Z}_2 \\ \mathbb{Z}_3 \end{pmatrix} \qquad \begin{pmatrix} a^\dagger \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \propto \begin{pmatrix} \overline{\mathbb{Z}}_0 \\ \overline{\mathbb{Z}}_1 \\ \overline{\mathbb{Z}}_2 \\ \overline{\mathbb{Z}}_3 \end{pmatrix} \tag{7.37}$$

This choice is consistent with the identification of Fock generators in Chapter 6. The generator $\mathbb{Z}_0$ is identified with a creation operator, whereas the generators $\mathbb{Z}_i$

are associated with annihilation operators (and conversely for the $\overline{Z}_\mu$ generators). The constant of proportionality is chosen so that:

$$\left[a, a^\dagger\right] = 1, \qquad \left[b_i, b_j^\dagger\right] = \delta_{ij}$$

From Equation 7.27, it can be seen that this constant of proportionality is equal to $(\hbar\lambda_0\pi_\theta)^{-\frac{1}{2}}$, where π_θ is the eigenvalue of Π_θ:

$$\Pi_\theta \left|n\right\rangle = \pi_\theta \left|n\right\rangle = \hbar(\ell + \lambda) \left|n\right\rangle .$$

This is true assuming the product $\lambda_0\pi_\theta > 0$. If this product is less than zero, then the Fock generators would need to be redefined. From now on it will be assumed that the product is greater than zero.

Explicitly then:

$$\mathbb{Z}_0 = \sqrt{\hbar\lambda_0\pi_\theta}\; a \qquad\qquad \overline{\mathbb{Z}}_0 = \sqrt{\hbar\lambda_0\pi_\theta}\; a^\dagger$$
$$\mathbb{Z}_1 = \sqrt{\hbar\lambda_0\pi_\theta}\; b_i^\dagger \qquad\qquad \overline{\mathbb{Z}}_0 = \sqrt{\hbar\lambda_0\pi_\theta}\; b_i.$$

Since:

$$\mathbb{X}^\mu = \frac{1}{\sqrt{2}}\left(\mathbb{Z}^\mu + \overline{\mathbb{Z}}^\mu\right) \qquad \mathbb{P}^\mu = \frac{i}{\sqrt{2}}\left(\overline{\mathbb{Z}}^\mu - \mathbb{Z}^\mu\right);$$

in terms of Fock operators the $\mathbb{X}_\mu$, $\mathbb{P}_\mu$ operators can be defined as:

$$\mathbb{X}_0 = \sqrt{\frac{\hbar\lambda_0\pi_\theta}{2}}\left(a + a^\dagger\right) \qquad\qquad \mathbb{X}_i = \sqrt{\frac{\hbar\lambda_0\pi_\theta}{2}}\left(b_i^\dagger + b_i\right)$$
$$\mathbb{P}_0 = i\sqrt{\frac{\hbar\lambda_0\pi_\theta}{2}}\left(a^\dagger - a\right) \qquad\qquad \mathbb{P}_i = i\sqrt{\frac{\hbar\lambda_0\pi_\theta}{2}}\left(b_i - b_i^\dagger\right).$$

The angular momentum and reciprocal boosts can also be written in terms of Fock operators. However, it is easiest to do this by breaking up the tensors $L^{\mu\nu}$ and $M^{\mu\nu}$ into constituent parts. The angular momentum can be broken up into pure boosts and pure rotations, by defining Lorentz boosts as:

$$K^i := L^{0i} = -L^{i0}.$$

A pure rotation is then given by

$$L^i := L^{jk} = -L^{kj},$$

where ijk are cyclical; for example: $ijk = 123$. Similarly, the reciprocal boosts can be broken up as follows:

$$U := M^{00} \qquad N^i := M^{0i} = M^{i0},$$

with the remaining elements labelled by two roman indices: M^{ij}. In terms of Fock operators:

$$\begin{aligned}
K^i &= \frac{1}{\lambda_0\pi_\theta}\left(\mathcal{X}^0\mathcal{P}^i - \mathcal{X}^i\mathcal{P}^0\right) - i\hbar\left(ab_i - a^\dagger b_i^\dagger\right)\\
L^i &= \frac{1}{\lambda_0\pi_\theta}\left(\mathcal{X}^j\mathcal{P}^k - \mathcal{X}^k\mathcal{P}^j\right) - i\hbar\left(b_j^\dagger b_k - b_k^\dagger b_j\right)\\
U &= \frac{1}{\lambda_0\pi_\theta}\left(\left(\mathcal{X}^0\right)^2 + \left(\mathcal{P}^0\right)^2\right) - \hbar\left(2a^\dagger a + 1\right)\\
N^i &= \frac{1}{\lambda_0\pi_\theta}\left(\mathcal{X}^0\mathcal{X}^i + \mathcal{P}^i\mathcal{P}^0\right) - \hbar\left(ab_i + a^\dagger b_i^\dagger\right)\\
M^{ij} &= \frac{1}{\lambda_0\pi_\theta}\left(\mathcal{X}^i\mathcal{X}^j + \mathcal{P}^j\mathcal{P}^i\right) - \hbar\left(b_i^\dagger b_j + b_j^\dagger b_i + \delta_{ij}\right).
\end{aligned}$$

The canonical Hamiltonian of Equation 7.28 can be written in terms of Fock operators as:

$$H_0 = N_0\lambda_0\pi_\theta\left(-a^\dagger a + b_1^\dagger b_1 + b_2^\dagger b_2 + b_3^\dagger b_3\right) + \frac{N_0\lambda_0}{2\alpha}{\pi_\theta}^2.$$

Acting on the number state $|n\rangle \equiv |n_0, n_1, n_2, n_3; \ell\rangle = |n_0, n_1, n_2, n_3\rangle \otimes |\ell\rangle$:

$$H_0\,|n\rangle = \left[N_0\lambda_0\hbar(\ell+\lambda)(-n_0+n_1+n_2+n_3) + \frac{N_0\lambda_0}{2\alpha}\hbar^2(\ell+\lambda)^2\right]|n\rangle\,.$$

7.2.2.1 Imposition of First Class Constraint

Before the constraint is imposed, the full quantum space of states is spanned by the basis (where the one dimensional space associated with the θ momentum has been omitted):

$$|p^\mu\rangle \otimes |n_0, n_1, n_2, n_3\rangle \equiv |p^\mu; n_0, n_1, n_2, n_3\rangle\,.$$

The constraint to be imposed is the operator condition that:

$$\tfrac{1}{2}(\mathbb{P}^\mu\mathbb{P}_\mu + \mathbb{X}^\mu\mathbb{X}_\mu) + \tfrac{1}{2}\Pi_\theta^2 = \Lambda. \tag{7.38}$$

Immediately, there are no restrictions on the energy-momentum operator $\mathcal{P}^\mu$ and so the mass spectrum is again the real line, producing timelike, massless and spacelike states. The imposition of the constraint therefore amounts to the projection of generic number states $|p^\mu; n_0, n_1, n_2, n_3\rangle$ onto a certain selected eigenspace of the Casimir operator of Equation 7.31: $\mathbb{P}^2 + \mathbb{X}^2$.

The following form of the constraint 7.38 will be used:

$$\begin{aligned}
&\tfrac{1}{2}(n_1 + n_2 + n_3 - n_0) + \tfrac{1}{2} = \Delta, \qquad (7.39)\\
&\text{where} \qquad \Delta := \tfrac{1}{2}\Lambda - \tfrac{1}{4}(n_\theta + \sigma)^2.
\end{aligned}$$

In this it has been assumed that a fixed eigenvalue $n_\theta + \sigma$ of the Π_θ momentum operator has been selected, for integral n_θ and up to a modular parameter σ, where

$0 \leq \sigma < 1$. The energy-momentum vector is not restricted so that again the mass spectrum is the whole real line, producing timelike states, massless states and spacelike states. From the form of the total angular momentum generator given in Equation 7.14, $\mathbb{L}^{\mu\nu} := \mathbb{X}^{\mu}\mathbb{P}^{\nu} - \mathbb{X}^{\nu}\mathbb{P}^{\mu}$ is identified with spin. The physical state space is therefore refined by diagonalising the second Poincaré group Casimir, which is defined in terms of the Pauli-Lubanksi vector and so corresponds to the spin quantum number (cf. Section 2.8.4.1 and Equation 2.108). The results are discussed in some detail in [43]. In the present work however, the results of the timelike states and the massless states are briefly discussed in Sections 7.2.2.2 and 7.2.2.3 respectively. In each case there is a little algebra, which is the Lie algebra of the usual Wigner little groups (cf. Sections 2.3.1 and 2.8.3). There is also a dual algebra or "commutant" algebra $\mathbb{L}(\hat{p})$ where $\hat{p}$ is a representative vector for each case (again, see Section 2.8.3 for more on the representative vectors) within the full enveloping algebra of the auxiliary Heisenberg algebra. This dual algebra commutes with the little algebra, and it controls the degeneracy of the unitary irreducible representations of the little algebra subject to the constraint 7.39.

The generators of $\mathcal{S}p^{(\mu)}(2,\mathbb{R}) \cong \mathcal{SU}(1,1) \cong \mathcal{SO}(1,2)$ are $K^{\mu}_{\pm}$ and K^{μ}_{0}. Here, the superscript μ refers to the Cartesian direction in Minkowski space ($\mu = 0,1,2,3$) with respect to which the respective symplectic algebra corresponds. Since there may be diagonal sums of several symplectic algebras, there may be several sets of these generators in any given dual algebra. These generators have commutation relations:

$$[K_0, K_{\pm}] = \pm K_{\pm}, \qquad [K_+, K_-] = -2K_0,$$

where the superscript has been dropped, and $K_+^{\dagger} = K_-$. The eigenvalues k of K_0 are integral (or they are half-integral for the spinor case) for unitary irreducible representations of $\mathcal{SO}(1,2)$. However, for general projective representations of $\mathcal{SU}(1,1)$, $k = E_0 + m$ for integral m and real E_0, where $0 \leq E_0 < 1$.

The quadratic $\mathcal{S}p(2,\mathbb{R})$ Casimir operator $\mathcal{C}_2$ is given by:

$$\begin{aligned} \mathcal{C}_2 :&= K_0^2 - \tfrac{1}{2}(K_+K_- + K_-K_+) \\ &= K_0^2 - K_0 - K_+K_-. \end{aligned}$$

This Casimir operator has eigenvalues $j(j+1)$. There are five series of representations:

- D_j^+ - the positive discrete series;
- D_j^- - the negative discrete series;
- $D(j)$ - the continuous series;
- D_0 - the trivial one dimensional series;

the fifth being the supplementary continues series, which is not needed here. The eigenvalue k is restricted in which values it can take, depending upon which series

representation it corresponds to. The possible values it can take are:

$$k = \begin{cases} -j, -j+1, -j+2, ... & \text{for } D_j^+; & -j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...; \\ j, j-1, j-2, ... & \text{for } D_j^-; & -j = 0, \frac{1}{2}, 1, \frac{3}{2}, ...; \\ 0, \pm 1, \pm 2, ... & \text{for } D(j); & j = -\frac{1}{2} + is,\ 0 \leq s < \infty. \end{cases}$$

Therefore for the positive (negative) discrete series representations, j is the minimal (maximal) eigenvalue of k and $j < 0$.

The rules for the decomposition of the tensor products for two discrete series of the same type are given by:

$$D_{j_1}^{\pm} \otimes D_{j_2}^{\pm} \to \sum_{j=j_1+j_2}^{\infty} \oplus D_j^{\pm}.$$

For two discrete series of opposite type are given by:

$$\text{(for } |j_1| > |j_2|) \quad D_{j_1}^+ \otimes D_{j_2}^- \to (D_{j_1-j_2}^+ \oplus D_{j_1-j_2+1}^+ \oplus ...) \oplus \int ds D(-\tfrac{1}{2} + is),$$

$$\text{(for } |j_2| > |j_1|) \quad D_{j_1}^+ \otimes D_{j_2}^- \to (D_{j_2-j_1}^- \oplus D_{j_2-j_1+1}^- \oplus ...) \oplus \int ds D(-\tfrac{1}{2} + is),$$

where the last term of the discrete series (the $(m+1)$th term) is $D_{j_1-j_2+m}^+$ or $D_{j_2-j_1+m}^-$, such that $-1 \leq \pm j_1 \mp j_2 + m < 0$ respectively, for $\mathcal{SO}(1,2)$ this is just -1 or $-\frac{1}{2}$. In the degenerate case where $j_1 = j_2 \equiv j$, the tensor product $D_j^+ \otimes D_j^-$ is simply given by the integral over the continuous series together with a copy of the trivial representation D_0.

7.2.2.2 Timelike States

See section 2.8.3.1 for further discussion on the timelike case of the Poincaré group. A suitable representative vector $\hat{p}$ is given by (up to dimensional considerations) $(1,0,0,0)$. The little group is $\mathcal{SO}(3)$. The dual algebra is given by $\mathcal{S}p^{(0)}(2,\mathbb{R}) + \mathcal{S}p^{(123)}(2,\mathbb{R})$ where $\mathcal{S}p^{(0)}(2\mathbb{R})$ is a symplectic algebra corresponding to the time coordinate, and $\mathcal{S}p^{(123)}(2,\mathbb{R})$ is a diagonal sum of three symplectic algebras; one for each space coordinate. In the timelike case, $p \cdot p = mc^2 > 0$ and there is an infinite series of integer-spin particles $\ell = 0, 1, 2, ...$ where ℓ is the eigenvalue of the $\mathcal{SO}(3)$ Casimir operator (the eigenvalue actually given by $\ell(\ell+1)$). In the reducible representation $D_{-1/4}^- \oplus D_{-3/4}^-$, states are spanned by the eigenvectors of $K_0^{(0)}$ with spectrum $-\frac{1}{4} - \frac{1}{2}k^{(0)}$, $k^{(0)} = 0, 1, 2$. For fixed ℓ, there are $K_0^{(123)}$ eigenvalues of the form $\frac{1}{2}\ell + \frac{1}{4} + k^{(123)}$, $k^{(123)} = 0, 1, 2, ...$ The constraint is therefore given by:

$$\begin{aligned} (\tfrac{1}{2}\ell + \tfrac{1}{4} + k^{(123)}) + (-\tfrac{1}{4} - \tfrac{1}{2}k^{(0)}) &= \Delta \\ \Rightarrow \qquad k^{(123)} - \tfrac{1}{2}k^{(0)} &= \Delta - \tfrac{1}{2}\ell. \end{aligned}$$

There is an infinite tower of occurrences of each ℓ. These are associated with even or odd modes of the $\mathcal{S}p^{(0)}(2,\mathbb{R})$ oscillator depending upon whether $\Delta - \frac{1}{2}\ell$ is

integral or half-integral. The eigenvalue $k^{(0)}$ of $K_0^{(0)}$ is fixed by the choice of $k^{(123)}$ so that a weight diagram of $[\ell]$ versus $k^{(123)}$ corresponds to the orbital angular momentum content of a three-dimensional isotropic simple harmonic oscillator system [43].

7.2.2.3 Massless States

See Section 2.8.3.2 for further discussion on the massless (lightlike) case of the Poincaré group. A suitable representative vector $\hat{p}$ is given by (up to dimensional considerations) $(1, 0, 0, 1)$. The little group is $\mathcal{E}(2)$ - the Euclidean group in 2 dimensions. The dual algebra is given by $\mathbb{E}(2)$ - another Euclidean group in 2 dimensions. The unitary irreducible representations for massless states are of continuous spin type, with a decomposition over continuous $\mathcal{E}(2)$ series representations. They have a Euclidean momentum of length π, where $0 < \pi < \infty$, with arbitrary helicity $(0, \pm 1, \pm 2, ...)$. For fixed π there is a further continuum of unitary irreducible representations of the dual algebra up to a re-scaling of the label of the dual momentum. Within each such representation the constraint simply selects the eigenvalue Δ of the diagonal generator [43].

Chapter 8

Conclusion

Several aspects of the symmetry of Born reciprocity were studied in this thesis. The group theory of Chapter 2 was modified in order to produce a group which displays the symmetry of Born reciprocity; the quaplectic group $\mathcal{Q}(1,3)$ - studied in Chapter 3 along with other introductory remarks about reciprocity. Chapter 4 introduced the concept of an İnönü-Wigner contraction and proceeded to apply several contractions to the quaplectic group for different limits. In Chapter 5 the state labelling problem for the compact group $\mathcal{Q}(2)$ was considered along with the reduction problem to $\mathcal{E}(2)$. In Chapter 6 the Schrödinger-Robertson inequality was modified in order to incorporate Born reciprocity. And in Chapter 7, the worldline quantisation of a simple particle-like system displaying quaplectic symmetry was considered.

In Chapter 2, several group theoretical concepts were introduced in order to be used throughout the rest of the thesis. In particular, the Mackey method of induced representations for semidirect product groups was briefly summarised. A few important symmetry groups were then studied in some detail, including the (abelian) translation group in Section 2.5, the Weyl-Heisenberg group in Section 2.6, the Lorentz group in Section 2.7 and the Poincaré group in Section 2.8. The Mackey method was applied to two of the Wigner little groups of the Poincaré group: the timelike case and the massless or lightlike case. Finally, the Klein-Gordon equation was derived from the eigenvalue equation of the unitary irreducible representations of the first Casimir operator $P^\mu P_\mu$ of the central extension of the Poincaré group. All of these concepts are well known in physics, and so it is difficult to see any new avenues of study along the lines of this chapter.

In Chapter 3, Born's theory of reciprocity of elementary particles was introduced and summarised. The symmetry of Born reciprocity were then generalised to form a group, called the quaplectic group $\mathcal{Q}(1,n)$ on n-phase space. Considerable work has been done on this group and its representations by Stephen Low. This group intrinsically displays the symmetry of quantum mechanics - namely the Weyl-Heisenberg group and the canonical commutation relation. There is also a new line element, defined in 1-phase space as:

$$ds^2 = -dt^2 + \frac{1}{c^2}dx^2 + \frac{1}{b^2}(dp^2 - \frac{1}{c^2}de^2),$$

with an obvious generalisation to n-phase space. The constant b is a new fundamental constant; the maximum rate of change of momentum. Thus it is the momentum analogue to the maximum rate of change of position - the speed of light - for space. The quaplectic representations and the Lie algebra were discussed and the Casimir operators defined. The eigenvalue equations of the unitary irreducible representations of the Casimir operators lead to the group's field equations and this process has been begun by Low [56]. It would be interesting to see these field equations fully defined, and so this is a definite area of possible further research. It would be particularly interesting to see if these field equations contract to the Poincaré field equations - the Klein-Gordon equation, Maxwell's equations, the Dirac equation and so on - in the inertial non-highly-interacting limit. If they do then this would be a significant step in the verification of the existence of Born reciprocity in reality. Chapter 3 continued by studying the relativity associated with the quaplectic group, called "reciprocal relativity". This involves a generalised version of the Lorentz factor γ, denoted by γ_{vfr} and defined as:

$$\gamma_{vfr} = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2 - \left(\frac{f}{b}\right)^2 + \left(\frac{r}{bc}\right)^2}}.$$

There are also modifications to the concepts of time dilation and length contraction. An example was studied for the case of time dilation where two observers are moving at a constant acceleration with respect to one another. In this situation, the dilated time t' is given by:

$$t' = \gamma_f \frac{c}{t} \arctan\left(\frac{\frac{at}{c}}{\sqrt{1 - (\frac{at}{c})^2}}\right),$$

where γ_f is the limit of γ_{vfr} as $v, r \to 0$. This formula could, for example, be applied to the orbit of the Pioneer spacecraft in order to determine whether or not the Pioneer anomaly is due to a reciprocal correction. If so this would provide a very powerful verification of Born reciprocity. Chapter 3 concludes with a brief summary of how proper acceleration or Rindler acceleration works in the quaplectic framework.

There are additional aspects of the principle of Born reciprocity which have not been studied in this work which could possibly lead to the verification of the principle. For example, Lashmar and Delbourgo in [50] and [25] investigated a Born reciprocal modification to a $1/r$ potential both on the large scale of galaxies and the small scale of atomic physics. This model suggests it might be possible to use Born reciprocity to solve the galaxy rotation problem without resorting to dark matter. However, the model studied had significant drawbacks, and would need much more work in order to have any confidence in it.

Chapter 4 began with a review of İnönü-Wigner contractions, and the particular case of the Poincaré group in 1 spatial dimension contracting to the Galilean group in the limit $c \to \infty$ following [40] was studied in order to explain the method. Another introductory example was that of the Weyl-Heisenberg group

contracting to the translation group in the limit $\hbar \to 0$. The chapter then proceeded to study several different limits for the quaplectic group, following mostly unpublished work by Low in addition to [58]. The limit $c \to \infty$ was taken in the matrix representations of the quaplectic group, which resulted in the Hamilton group. The Hamilton group, due to Low in [61, 59], is the group of inertial classical mechanics. This group itself is worthy of further study. The normal subgroup of the central extension of the inhomogeneous Hamilton group is the Heisenberg group. The centrally extended group has the Galilean central charge M and the Heisenberg central charge I but it also has a new central charge A, which has the dimensions of reciprocal tension. The particular case of the inertial Hamilton group (setting the relative force f and power r to zero) is the Galilean group.

Another example of a contraction studied was the important example of the quaplectic group contracting to the Poincaré group. This is done by setting the relative force f and power r to zero and taking the limit $b \to \infty$. In other words, it is the inertial case of the quaplectic group and the contraction takes the mechanics from highly-interacting reference frames to non-highly-interacting reference frames. This group too can be contracted to the Galilean group in the limit $c \to \infty$. A third path exists from the quaplectic group to the Galilean group whereby the relative velocity v and power r are set to zero and the limit $c \to \infty$ is taken as an intermediate stage, before taking the limit $b \to \infty$. Therefore, the contractions commute.

An immediate extension to the work done in this chapter would be to carry out the contractions on 3-phase space or n-phase space. However, there is not too much added complexity in these cases, and the contracted generators and commutation relations can be taken direct from the commutation relations of the n-phase space generators of the quaplectic group defined in Section 3.1.3.1. Additionally, the mechanics of the intermediate group reached by setting $v = r = 0$ and taking the limit $c \to \infty$ could also be examined. Finally, the field equations for each group mentioned should contract to the field equations of the contracted group when the appropriate limits are taken. Once the field equations for the quaplectic group have been properly established, this would be an obvious exercise to undertake.

In Chapter 5 the $\mathcal{Q}(2) \supset \mathcal{E}(2)$ reduction problem and state labelling problem is studied for the scalar case. Circular creation and annihilation operators are used in order to set up basis states in which the $\mathcal{O}(2)$ operator M is diagonal. Basis states $\sum_{r=0}^{\infty} \lambda_r^+(p^2, m) \, |r + 2m, r\rangle$ are set up and the coefficients λ_r^+ determined. The momentum wavefunction is then reconstructed in the position representation, with the result that the states are given by Bessel functions. Several operators are then checked and evaluated, in order to determine the form of the quaplectic generators $Z'_\pm$ and U.

There is much more work that could be done on this topic. The commutation relations for the quaplectic group could be checked in order to determine the generators are in the correct form. Additionally, only the scalar case has been examined so the addition of spin would be a worthy avenue of investigation. Finally, once the framework has been completely verified, the same calculations should be done for the quaplectic group in 1 spatial dimension and the reduction problem $\mathcal{Q}(1,1) \supset \mathcal{E}(1,1)$.

In Chapter 6, the Schrödinger-Robertson inequality was adapted to the scalar case of quaplectic symmetry. The inequality becomes:

$$\det(\Sigma) \geq (\tfrac{1}{2}\hbar)^8,$$

where Σ is the covariance matrix. The quaplectic invariance of $\det(\Sigma)$ was also established, and for uncorrelated states a natural extension of the Heisenberg uncertainty principle was found, such that:

$$\Delta X^0 \Delta X^1 \Delta X^2 \Delta X^3 \Delta P^0 \Delta P^1 \Delta P^2 \Delta P^3 \geq (\tfrac{1}{2}\hbar)^4.$$

It was also found that definite attributes $Q_{\mu\nu} := \langle X_\mu X_\nu \rangle$, $T_{\mu\nu} := \langle P_\mu P_\nu \rangle$ and $R_{\mu\nu} := \langle X_\mu P_\nu + X_\nu P_\mu \rangle$ could be associated to the state of the quaplectic system. Finally, it was concluded that different (but unitarily equivalent) minimal uncertainty states correspond to physically distinct semiclassical limits only if the unitary transformation relating them does *not* belong to the quaplectic group. It is anticipated that the quantum aspects of the results of this chapter act at the Planck scale. It is possible that there are some Schrödinger-Robertson minimal uncertainty states which contain a particle spectrum, and others which may not have this structure. This is presently just speculation, but constitutes a promising new avenue of investigation.

In Chapter 7, the mathematical formalism of Hamiltonian quantisation and constraint mechanics was applied to a reciprocally invariant system, a system which displays the symmetries of the quaplectic group. In order to diagonalise the Hamiltonian, two different sets of Fock generators were used. The first choice maintained Lorentz covariance throughout all of the quantum states but produced negative norm states. By considering free motion on the Heisenberg group, a quaplectic-invariant action was constructed and coupled to a worldline in a diffeomorphic invariant manner. The spectrum of the Hamiltonian thus achieved is organised into Landau-like levels. In coupling to a worldline geometry there arose a constant parameter which plays the role of a worldline cosmological constant. An appropriate choice of this constant removes negative norm states from the physical spectrum but the remaining states are all of vanishing spin. There were no restrictions placed on the values of the energy-momentum vector, and so the physical spectrum of $p \cdot p$ consists of the whole real line. Therefore there are timelike, spacelike, massless and null states. It is possible that an investigation into a supersymmetric construction of the worldline and/or the spacetime geometry could lead to the exclusion of the tachyonic branch of the mass spectrum. However, in this case it is hard to see that the mass spectrum would not still be continuous - even if only for the timelike and massless cases. Another possibility is that a continuous mass spectrum does exist in reality, with the deeper symmetry of the quaplectic group displaying this reality. If this is to be true then there would have to be some mathematical reason whereby the continuous states decompose into the discrete mass states that we observe. A possible candidate for this decomposition could be the limit $b \to \infty$. This would mean that in highly-interacting fast-moving reference frames, particles lose their discrete mass. An obvious corollary is that the particles only gain discrete mass levels when observed from a relative non-highly-interacting and/or non-fast-moving reference frame.

The second set of Fock generators had no negative norm states but sacrificed Lorentz covariance. Imposition of the first class constraint places no restrictions on the energy-momentum vector, so that again the mass spectrum of $p \cdot p$ is the real line. The cases of the timelike branch and the massless branch are summarised. The physical state space is refined by the diagonalisation of the second Poincaré group Casimir operator. Therefore, as opposed to the Lorentz covariant choice of Fock generators, the existence of spin is explicitly included in the physical states. For the timelike branch there is an infinite number of particles with spins $\ell = 0, 1, 2, ...$ each of which is countably degenerate. The spin content as a whole is therefore equivalent to the orbital angular momentum decomposition of a nonrelativistic isotropic 3 dimensional simple harmonic oscillator. For the massless states, the "particles" are of continuous spin type and arbitrary integer helicity, and each such non-minimal massless representation is continuously degenerate. The spacelike branch is not summarised in this work, however from [43] it is possible that the spacelike states may be at least partially eliminated by a careful choice of Δ.

There are several possible avenues for further research on this topic, which could possibly eliminate some of the technical difficulties. The geometry of the coordinates $x^\mu(\tau)$, $p^\mu(\tau)$, $\theta(\tau)$ consists of a "flat" target space, and so it is possible that the continuous spectrum of $p \cdot p$ may be eliminated by an appropriate choice of geometry. A supersymmetric modification could also lead to promising results. Finally, it is possible again that the unconventional particle states could correspond to genuine exotic matter and so should be studied seriously.

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